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A NOTE ON THE EXPONENTIAL BOUNDS FOR SEQUENCES OF LONG-RANGE DEPENDENCE

BY

HWAI-CHUNG HO

Dedicated to the memory of Professor Tsing-Houa Teng

Abstract. Certain type of exponential inequalities are derived for the sample mean $(\sum_{i=1}^{n} G(X_i))/n$, where $\{X_i\}$ is a stationary sequence of standard Gaussian random variables which may dispaly long-range dependence, and $G(\cdot)$ is some non-linear Borel function.

1. Introduction

Let $\{Y_j, j = 1, 2, \ldots\}$ be a sequence of iid random variables with finite expectation $\mu = EY_j$. Many asymptotic properties, especially those of strong type, concerning the n-th sample mean $\bar{S}_n = (\sum_{j=1}^{n} Y_j)/n$ are frequently derived by making use of the following type of exponential inequalities

$$P(|\bar{S}_n - \mu| \geq \varepsilon) \leq A \exp\{-ng(\varepsilon)\},\ \varepsilon > 0, \tag{1.1}$$

where A is a positive constant and $g(\cdot)$ is a positive function whose explicit form may depend on the specific distribution of Y_j; the validity of (1.1) usually requires Y_j to be bounded. A number of well-known inequalities, e.g., Bernstein's, Hoeffding's and Bennett's, are included in (1.1) (see Shorack and Wellner, 1986, Appendix A).

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When $\{Y_j\}$ is a strictly stationary sequence, bounds similar to (1.1) can still be obtained under various type of mixing conditions (see, e.g., Györfi, Härdle, Sarda, and Vieu, 1989, Ch.1 and references therein). In this note we will establish probability bounds like (1.1) under the dependence structure that allows the sequence $\{Y_j\}$ to violate even the strongly mixing condition. In Section 2, two exponential inequalities are obtained and stated separatedly in Theorem 1 and Theorem 2. The latter is formulated as a special case of the former and includes the class of so-called long-range dependent processes. Proofs of the theorems are detailed in Section 3.

2. Statement of assumptions and results

Throughout this paper we concentrate on the sequence $\{Y_j\}$ which is modeled by

$$Y_j = G(X_j), \qquad j = 1, 2, \ldots.$$

$G(\cdot)$ is a non-linear Borel function and $\{X_j\}$ is a stationary sequence of standard Gaussian random variables. Let $r(n) = EX_jX_{j+n}$ be the n-lag covariance of $\{X_j\}$ and let

$$r^*(n) = \sup_{|n|\le m} |r(m)| \tag{2.1}$$

For convenience of later discussion we introduce an auxiliary iid sequence $\{Y_j'\}$ whose marginal distribution is the same as that of $Y_j = G(X_j)$, and denote by $\bar{S}_n' = (\sum_{j=1}^n Y_j')/n$ the n-th sample mean of $\{Y_j'\}$. Following two assumptions will be needed.

(A1). (1.1) holds for $\bar{S}_n'$.

(A2). There exists a non-decreasing sequence of positive integers $\{d_n\}$ such that $d_n \le n$ and

$$(r^*(d_n)n/d_n) + (n/d_n)^{-1}\log d_n = o(1). \tag{2.2}$$

We now formulate

Theorem 1. *Suppose* (A1) *and* (A2) *holds. Then for every* $\xi > 0$, *there* *exists* $n'(\xi)$ *such that* $n \geq n'$

$$P(|\bar{S}_n - \mu| \geq \xi) \leq \exp\{-B(n/d_n)g(\xi)\}, \tag{2.3}$$

where $\mu = EY_j, B > 0$ *is some constant, and* d_n *is as specified in* $(A2)$.

Remark 1. As an illustration of Theorem 1, a class of covariance functions is given below to see when the assumption (A2) can be met. Suppose the covariance function $r(n)$ of the underlying Gaussian process $\{X_j\}$ is dominated by a regularly varying function as

$$|r(n)| \leq |n|^{-\alpha} L(n), \qquad 0 < \alpha, \tag{2.4}$$

where $L(x)$ is a function varying slowly at $+\infty$, i.e., $L(x) > 0$ and $\lim_{n\to\infty} L(cx)/L(x) = 1$ for all $c > 0$. Define

$$\sup_{m \geq |n|} m^{-\alpha} L(m) = u(n).$$

Then, by (2.4), $r^*(n) \leq u(n)$, and the function $u(n)$ is also regularly varying with index -α (Seneta, 1976, p.20-21), i.e.,

$$u(n) = |n|^{-\alpha} L_0(n)$$

for some slowly varying $L_0(n)$ representated by (Seneta, 1976, Theorem 1.2)

$$L_0(x) = \exp\{\eta(x) + \int_U^x \frac{\varepsilon(t)}{t} dt\}, \tag{2.5}$$

where $\eta(x)$ is a bounded measurable function on $[U, \infty)$ such that $\lim_{n\to\infty} \eta(x) = c\,(|c| < \infty)$, and $\varepsilon(x)$ is a continuous function on $[U, \infty)$ such that $\lim_{x\to\infty} \varepsilon(x) = 0$. Set $\eta^* = \sup_x |\eta(x)|$ and define

$$\bar{L}_0(x) = \exp\{\eta^* + \int_U^x \frac{|\varepsilon(t)|}{t} dt\}.$$

Then $L_0(x)$ is also slowly varying at $+\infty$, and from (2.5)

$$0 < \bar{L}_0(x) \uparrow \quad \text{and} \quad L_0(x) \leq \bar{L}_0(x). \tag{2.6}$$

Choose $1 > \beta > (1+\alpha)^{-1}$ and

$$d_n = \min\{[n^\beta], n\} \tag{2.7}$$

where $[x]$ means the integral part of x. It is straightforward that $(n/d_n)^{-1}\log d_n = o(1)$, and, by (2.6),

$$r^*(d_n)n/d_n = o(n^{-\beta(1+\alpha)+1}\bar{L}_0(n)) = o(1),$$

which verifies (2.2).

Continuing Remark 1 gives

Theorem 2. *Assume* (2.4) *and let* $|Y_j - \mu| \le M < \infty$ *and Var* $Y_j = \sigma^2 > 0$. *Then for every* $\varepsilon > 0$ *there is* $n''(\varepsilon)$ *such that*

$$P(|\bar{S}_n - \mu| \ge \varepsilon) \le \exp\{-B'(\varepsilon, \sigma^2, M)\xi(n)\}, \tag{2.8}$$

for all $n \ge n''$, *where* $B'(\varepsilon, \sigma^2, M) = D \cdot \varepsilon^2/2(\sigma^2 + (\varepsilon M/3))$ *for some positive constant* D, *and* $\xi(n) = n^\lambda$ *with* $0 < \lambda < \alpha/(1+\alpha)$.

Consider a special case of (2.4) that the covariances are of the form

$$|n|^{-\alpha_1}L_1(n) \le |r(n)| \le |n|^{-\alpha_2}L_2(n), \qquad 0 < \alpha_2 \le \alpha_1 < 1, \tag{2.9}$$

where L_1 and L_2 are both slowly varying at infinity. One can have $\alpha_2 < \alpha_1$ in (2.9) when the spectral density of $\{X_j\}$ has multiple poles including one at the zero frequencey (Dobrushin and Major, 1979, Remark 4.2). The process which staisfies (2.9) with $r(n) = |n|^{-\alpha}L(n)$, $0 < \alpha < 1$ and slowly varying $L(n)$, is often referred as process of long-rang dependence or strong dependence to reflect the fact that

$$\sum_{n=1}^{\infty} r(n) = +\infty$$

(see Cox, 1984, Taqqu, 1985, and Beran, 1992, for a review on long-range dependent processes). The best known example of long-range dependent process is the fractional Gaussian noise (Mandelbrot and van Ness, 1968), which is a stationary Gaussian process $\{Z_j, j = 1, 2, \ldots\}$ with covariances

$$\begin{aligned}\operatorname{cov}(Z_j, Z_{j+n}) =& \sigma_z^2(|n+1|^{2H} - 2|n|^{2H} + |n-1|^{2H})/2\\ \sim& \sigma_z^2 H(2H-1)|n|^{2H-2}\end{aligned}$$

where $\sigma_z^2 = \mathrm{Var} Z_j$, $1/2 < H < 1$, and "$\sim$" means asymptotically equivalent. The H is usually called self-similarity parameter as $\{Z_j\}$ can be constructed from the stationary increments of an H-self-similar Gaussian process. Note that due to the peristent dependence shown in (2.11) both of the sequences $\{X_j\}$ and $\{Y_j = G(X_j)\}$ may not be strongly mixing (Rosenblatt, 1961).

Remark 2. As $Y_j = G(X_j)$ is bounded, one has the representation

$$Y_j - \mu = \sum_{m=1}^{\infty} \frac{a_m}{m!} I_m(e^{ij(x_1+\ldots+x_m)}) \tag{2.10}$$

(Major, 1981, Theorem 6.1), where

$$I_m(f) = \int f(x_1,\ldots,x_m) Z(dx_1)\ldots Z(dx_m)$$

denotes the m-fold Wiener-Ito integral with respect to the random spectral measure $Z(\cdot)$ generated by $\{X_j\}$. Note that the expansion (2.10) must have infinitely many terms since Y_j is bounded. Mckean (1973) and Major (1981) obtain that for every positive integer $m \geq 1$ there exist $Q = Q(m)$ and $N = N(m)$ such that

$$P(|V| \geq x) \leq \exp\left\{-\frac{x^{2/m}}{Q}\right\}, \qquad x \geq N, \tag{2.11}$$

for all m-fold Wiener-Ito integral $V = I_m(f)$ with unit variance. The probability bound in (2.11) is sharp, whereas it deals only with one term in the expansion (2.10) and $N(m)$ needs to be as large as $m^{-1/4}(\sqrt{2/3}e)^{m/2}$ (Plikusas, 1981, Theorem 1). Under the setting of long-range dependence as specified by (2.9), the question on how to make a link between (2.8) and (2.11) still remains unanswered.

3. Proofs

Proof of Theorem 1. We first divide the centered sums into a number of disjoint blocks. To this end, set $Y_j^* = Y_i - \mu$ and for $1 \leq i \leq d_n$

$$n(i) = \begin{cases} [n/d_n] & \text{if } i + [n/d_n]d_n \leq n \\ [n/d_n] - 1 & \text{if } i + [n/d_n]d_n > n \end{cases}$$

Then

$$S_n^* = \sum_{j=1}^{n} Y_j^* = \sum_{i=1}^{d_n} \sum_{j=0}^{n(i)} (Y_{i+jd_n}^*) \equiv \sum_{i=1}^{d_n} S^*(n(i), d_n),$$

and

$$P(|\bar{S}_n - \mu| \geq \varepsilon) \leq \sum_{i=1}^{d_n} P(\frac{|S^*(n(i), d_n)|}{n(i)} \geq \varepsilon) \equiv \sum_{i=1}^{d_n} J(n, i, \varepsilon)$$

Recall from (2.1) that as $n \to \infty$

$$r^*(n) \downarrow 0. \tag{3.2}$$

For each $i, 1 \leq i \leq d_n$, consider the set of $n(i)$ Gaussian random variables $\{X_{i+jd_n}, j = 1, 2, \ldots, n(i)\}$, The covariance of any two of the random variables $X_i + j_1 d_n$ and $X_i + j_2 d_n, j_1 \neq j_2$, satisfies

$$\begin{aligned} n(i)|EX_{i+j_1 d_n} X_{i+j_2 d_n}| &= n(i)|r(|j_1 - j_2| d_n)| \\ &\leq r^*(d_n) n(i) \qquad \text{(by (3.2))} \\ &< 1, \text{for large } n \geq n_0 \qquad \text{(by 2.2)} \end{aligned} \tag{3.3}$$

This enables us to use Lemma 3.3 and Lemma 3.4 of Taqqu (1977) to express the joint probability $f_{n,n(i)}$ of $\{X_{i+jd_n}, j = 1, \ldots, n(i)\}$ as a uniformly convergent series over $R^{n(i)}$: For $n \geq n_0$

$$\begin{aligned} &f_{n,n(i)}(x_1, \ldots, x_{n(i)}) \\ &= \sum_{q=0}^{\infty} \sum_{(k_1, \ldots, k_{n(i)}, q)} \left\{ E \prod_{j=1}^{n(i)} H_{k_j}(X_{i+jd_n}) \right\} \prod_{j=1}^{n(i)} \frac{H_{k_j}(x_j)}{k_j!} \cdot \varphi(x_j), \end{aligned} \tag{3.4}$$

where we set

$$\sum_{(k_1, \ldots, k_{n(i)}, q)} \equiv \sum_{\substack{k_1 + \ldots + k_{n(i)} = 2q \\ 0 < k_1, \ldots, k_{n(i)} \leq q}}$$

and $H_k(x)$ denotes the k-th Hermite polynomial with leading coefficient one and $\varphi(x) = (2\pi)^{-1/2} \exp(-x^2/2)$ is the standard Gaussian density. Define

$$B(n, i, \varepsilon) = \{(x_1, \ldots, x_{n(i)}) | \sum_{j=1}^{n(i)} (G(x_j) - \mu) \geq n(i)\varepsilon\}.$$

An approximation of $J(n,i,\varepsilon)$ via (3.4) shows

$$\begin{aligned} J(n,i,\varepsilon) =& \sum_{q=0}^{\infty} \sum_{(k_1,\ldots,k_{n(i)},q)} \left\{ E \prod_{j=1}^{n(i)} H_{k_j}(X_{i+jd_n})/\sqrt{k_j!} \right\} \\ & \cdot \int_{B(n,i,\varepsilon)} \prod_{j=1}^{n(i)} \frac{H_{k_j}(x_j)}{\sqrt{k_j!}} \cdot \varphi(x_j)dx_j \\ \leq & \left\{ \sum_{q=0}^{\infty} \sum_{(k_1,\ldots,k_{n(i)},q)} |E \prod_{j=1}^{n(i)} \frac{H_{k_j}(X_{i+jd_n})}{\sqrt{k_j!}}| \right\} \\ & \cdot \left\{ \int_{B(n,i,\varepsilon)} \prod_{j=1}^{n(i)} \varphi(x_j)dx_j \right\}^{1/2} \qquad (3.5) \\ & \text{(by Holder's inequality and } \int H_k^2(x)\varphi(x)dx = k!) \\ \equiv & T_{i,1} \cdot T_{i,2}. \end{aligned}$$

(A1) implies

$$T_{i,2} \leq [Ae^{-n(i)g(\varepsilon)}]^{1/2} \qquad (3.6)$$

Set $V(n,i) = [r^*(d_n)(n(i)-1)]^{1/2}$, and apply Lemma 3.1 and Lemma 3.2 of Taqqu (1977) (cf. the proof of Proposition 3.1 of the paper) to obtatin

$$\begin{aligned} T_{i,1} \leq & \sum_{q=0}^{\infty} \sum_{(k_1,\ldots,k_{n(i)},q)} r^*(d_n)^{(k_1+\ldots+k_{n(i)})/2} \prod_{j=1}^{n(i)} (n(i)-1)^{k_j/2} \\ \leq & (\sum_{k=0}^{\infty} V^k(n,i))^{n(i)} \qquad \text{(by (3.3))} \qquad (3.7) \\ = & \exp(-n(i)\log(1-V(n,i))). \end{aligned}$$

Combining (3.6) and (3.7) gives

$$\max_{1\leq i\leq d_n} T_{i,1}\cdot T_{i,2} \leq \sqrt{A}\exp\{-([n/d_n]-1)[g(\varepsilon)/2+\log(1-\max_{1\leq i\leq d_n} V(n,i))]\} \quad (3.8).$$

Note by (2.2) that $\max_{1\leq i\leq d_n} V(n,i) \to 0$ as $n\to\infty$. Hence, from (3.1) and (3.8), there are $B>0$ and $n_1(\varepsilon)$ such that

$$P(|\bar{S}_n - \mu| \geq \varepsilon) \leq \exp\{-B(n/d_n)g(\varepsilon)\}$$

for all $n \geq \max\{n_0, n_1(\varepsilon)\}$. This completes the proof.

Proof of Theorem 2. Substituting (1.1) by the following simplified version of Bennett's inequality (Shorack and Wellner, 1986, p.851-852)

$$P(|\bar{S}_n' - \mu| \geq \varepsilon) \leq 2\exp\left\{-\frac{n\varepsilon^2}{2(\sigma^2 + (\varepsilon M/3))}\right\}.$$

We then see that (2.8) is immediate from Theorem 1 with d_n as given in (2.7).

References

[1] J. Beran, *Statisticalmethodsfordatawithlong›rangedependence*, Statistical Science V., 7(1992), 404-420.

[2] D.R. Cox, Long-range dependence: a review. In statistics: An Appraisal. Proceedings 50th Anniversary Conference, Ed. H.A. David and H.T. David, 55-74, The Iowa State University Press, 1984.

[3] R. L. Dobrushin, and P. Major, *Non›centrallimittheoremsfornonlinearfunctionalsofGaus›sionfields‹* Z. Wahrsch. Verw. Gebiete 50(1979), 27-52.

[4] L. Györfi, W. Härdle, P. Sarda and P. Vieu, *NonparametricCurveEstimationfromTime Series*, Lecture Notes in Statistics, 60 (1989), Springer-Verlag, New York-Berlin.

[5] P. Major, *MulitipleWiener›ItoIntegrals.WithApplicationstoLimitsTheorems*, Lecture Notes in Mathematics, 849 (1981), Springer-Verlag, New York-Berlin.

[6] H.P. Mckean, *Wienerφstheoryofnonlinearnoise*, Stochastic Differential Equations, SIAM-AMS Proc. 6(1973), 191-209.

[7] B.B. Mandelbrot, and J.W. Van Ness,*FractionalBrownianmotion‹fractionalnoiseandappli›cations‹* SIAM Review 10, 4(1968), 422-437.

[8] A. Plikusas, *SomepropertiesofthemultipleItointegral* (in Russian), Liet. Math. Rink. 21(1981), 163-173.

[9] M. Rosenblatt, *Independenceanddependence‹* Proc. Forth Berkeley Symp, Math. Statis. Probab., 431-443. Univ. California Press (1961).

[10] E. Seneta, Regularly Varying Functions, Lecture Notes in Mathematics, 508. Springer-Verlag, New York-Berlin (1976).

[11] G. Shorack and J. Wellner, Empirical Processes with Applications to Statistics, John Wiley, New York (1986).

[12] M.S. Taqqu, *Lawoftheiteratedlogarithmforsumsofnon›linearfunctionsofGaussianvariablesthat exhibitalongrangedependence‹* Z. Wahrsch. verw. Gebiete 40 (1977), 203-238.

[13] M.S. Taqqu,*Abibliographicalguidetoself›similarprocessesandlong›rangedependence‹* In Dependence in Probability and Statistics, Ed. E. Eberlein and M.S. Taqqu, pp. 137-165, Birkhauser, Basel (1985).

Institute of Statistical Science, Academia Sinica, Taipei, Taiwan, R.O.C.