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SOME INEQUALITIES OF HADAMARD TYPE

BY

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Abstract. A sharp Hadamard's inequality is proved for the class of functions introduced by Godunova and Levin. A new class $P(I)$ of quasi-convex functions on an interval I is introduced ($f : I \to R_+$ belongs to $P(I)$ iff, for all $x, y \in I$ and $\lambda \in [0,1]$, $f(\lambda x + (1-\lambda)y) \le f(x) + f(y)$) and a sharp Hadamard's inequality is proved for that class. The results obtained can e.g. be applied for the class of nonnegative monotone functions and some applications are pointed out.

1. Introducton

Let $I = [a_0, b_0]$ be an interval on the real line, let $f : I \to R$ be a convex function and let $a, b \in I$, $a < b$. We consider the well-known Hadamard's inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}.$$

Both inequalities hold in the reversed direction if f is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Some new informations connected to this inequality have recently been obtained in [1-5] (see also the new books [8-9]). In Section 1 of this paper we will prove some sharp integral

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inequalities of Hadamard type for a class of functions introduced and studied by Godunova and Levin [6] (for some discrete inequalities in this class of functions see [6-8]). We remark that this class of functions has the remarkable property that it is equivalent to the class of functions satisfying a generalized form of the Schur inequality (see below). In Section 3 we will introduce a new class of functions between the class of nonnegative convex functions and the class of Godunova-Levin and still containing all nonnegative convex, quasi-convex and monotone functions. We prove a Hadamard type inequality in this class of functions and point out the fact that this result is sharp. This version of the Hadamard inequality is useful for applications (see e.g. our Example 4.1). This fact together with some other concluding remarks are pointed out in our Section 4.

2. Hadamard's Inequality for The Godunova-Levin Class of Functions

In 1985 E.K. Godunova and V.I. Levin [6] (see also [7] or [8, pp. 410-433]) introduced the following remarkable class of functions:

A map $f : I \to R$ is said to belong to the class $Q(I)$ if it is nonnegative and, for all $x, y \in I$ and $\lambda \in (0,1)$, satisfies the inequality

$$f(\lambda x + (1-\lambda)y) \leq \frac{f(x)}{\lambda} + \frac{f(y)}{1-\lambda}. \tag{2.1}$$

They also noted that all nonnegative monotone and nonnegative convex functions belong to this class and they also proved the following motivating result:

If $f \in Q(I)$ and $x, y, z \in I$, then

$$f(x)(x-y)(x-z) + f(y)(y-x)(y-z) + f(z)(z-x)(z-y) \geq 0. \qquad (2.1)'$$

In fact, $(2.1)'$ is even equivalent to (2.1) so it can alternatively be used in the definition of the class $Q(I)$ (see [7]).

For the case $f(x) = x^r$, $r \in R$, the inequality $(2.1)'$ obviously coincides with the well-known Schur inequality.

Our main result in this section reads:

Theorem 2.1. *Let $f \in Q(I)$, $a, b \in I$, with $a < b$ and $f \in L_1[a,b]$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{4}{b-a}\int_a^b f(x)dx, \tag{2.2}$$

and

$$\frac{1}{b-a}\int_a^b p(x)f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{2.3}$$

where $p(x) = \dfrac{(b-x)(x-a)}{(b-a)^2}$, $x \in I$.

The constant 4 in (2.2) is the best possible.

Proof. Since $f \in Q(I)$ we have for all $x, y \in I$ (with $\lambda = 1/2$ in (2.1))

$$2(f(x)+f(y)) \geq f\left(\frac{x+y}{2}\right),$$

i.e., with $x = ta + (1-t)b$, $y = (1-t)a + tb$,

$$2(f(ta+(1-t)b) + f((1-t)a+tb)) \geq f\left(\frac{a+b}{2}\right).$$

By integrating we therefore have that

$$2(\int_0^1 f(ta+(1-t)b)dt + \int_0^1 f((1-t)a+tb)dt) \geq f\left(\frac{a+b}{2}\right). \tag{2.4}$$

Since

$$\int_0^1 f(ta+(1-t)b)dt = \int_0^1 f((1-t)a+tb)dt = \frac{1}{b-a}\int_a^b f(x)dx,$$

we get the inequality (2.2) from (2.4).

For the proof of (2.3) we first note that if $f \in Q(I)$, then, for all $a, b \in I$ and $\lambda \in [0,1]$, it yields

$$\lambda(1-\lambda)f(\lambda a + (1-\lambda)b) \leq (1-\lambda)f(a) + \lambda f(b),$$

and

$$\lambda(1-\lambda)f((1-\lambda)a + \lambda b) \leq \lambda f(a) + (1-\lambda)f(b).$$

By adding these inequalities and integrating we find that

$$\int_0^1 \lambda(1-\lambda)(f(\lambda a+(1-\lambda)b)+f((1-\lambda)a+\lambda b))d\lambda \le f(a)+f(b). \quad (2.5)$$

Moreover,

$$\begin{aligned}\lambda(1-\lambda)\int_0^1 f(\lambda a+(1-\lambda))d\lambda =&\lambda(1-\lambda)\int_0^1 f((1-\lambda)a+\lambda b)d\lambda\\ =&\frac{1}{b-a}\int_a^b \frac{(b-x)(x-a)}{(b-a)^2}f(x)dx. \quad (2.6)\end{aligned}$$

We get (2.3) by combining (2.5) with (2.6) and the proof is complete. The constant 4 in (2.2) is the best possible because this inequality obviously reduces to an equality for the function

$$f(x)=\begin{cases}1, & a\le x<\frac{a+b}{2},\\ 4, & x=\frac{a+b}{2},\\ 1, & \frac{a+b}{2}<x\le b.\end{cases}$$

Moreover, this function is of the class $Q(I)$ because

$$\frac{f(x)}{\lambda}+\frac{f(y)}{1-\lambda}\ge\frac{1}{\lambda}+\frac{1}{1-\lambda}=g(\lambda)\ge\min_{0<\lambda<1}g(\lambda)=g(\frac{1}{2})=4\ge f(\lambda x+(1-\lambda)y).$$

The proof is complete.

3. Hadamard's Inequality for a New Class of Functions

In this section we will restrict the Godunova-Levin class of functions and prove a sharp version of Hadamard's inequality in this class. More precisely, we say that a map $f:I\to R$, belongs to the class $P(I)$ if it is nonnegative and, for all $x,y\in I$ and $\lambda\in[0,1]$, satisfies the following inequality

$$f(\lambda x+(1-\lambda)y)\le f(x)+f(y). \quad (3.1)$$

Obviously, $Q(I)\supset P(I)$ and for applications it is important to note that also $P(I)$ contain all nonnegative monotone, convex and quasi-convex functions, i.e., nonnegative functions satisfying

$$f(\lambda x+(1-\lambda)y)\le\max(f(x),f(y)).$$

Theorem 3.1. *Let* $f \in P(I)$, $a, b \in I$, *with* $a < b$ *and* $f \in L_1[a,b]$. *Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a}\int_a^b f(x)dx \leq 2(f(a)+f(b)). \tag{3.2}$$

Both inequalities are the best possible.

Proof. According to (3.1) with $x = ta + (1-t)b$, $y = (1-t)a + tb$ and $\lambda = 1/2$ we find that

$$f\left(\frac{a+b}{2}\right) \leq f(at+(1-t)b) + f((1-t)a+tb).$$

Thus, by integrating, we obtain

$$f\left(\frac{a+b}{2}\right) \leq \int_0^1 (f(at+(1-t)b) + f((1-t)a+tb))dt = \frac{2}{b-a}\int_a^b f(x)dx,$$

and the first inequality is proved. The proof of the second inequality follows by using (3.1) with $x = a$ and $y = b$ and integrating with respect to λ over $[0,1]$.

The first inequality in (3.2) reduces to an equality for the (nondecreasing) function

$$f(x) = \begin{cases} 0, & a \leq x < \frac{a+b}{2}, \\ 1, & \frac{a+b}{2} \leq x \leq b, \end{cases}$$

and the second inequality reduces to an equality for the (nondecreasing) function

$$f(x) = \begin{cases} 0, & x = a, \\ 1, & a < x \leq b. \end{cases}$$

The proof is complete.

4. An Example and Concluding Remarks

4.1. Example. Let $f(x) = x^p$, $0 < p \leq 1$, $0 \leq x \leq \infty$. By using Hadamard's inequality for this (concave) functions we find that, for every a, b, $b > a$,

$$\left(\frac{a+b}{2}\right) \geq \frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)} \geq \frac{a^p+b^p}{2}.$$

Obviously, $f(x) \in P(I)$ and, thus, we can apply our Theorem 3.1 to see that also the following reversed inequalities hold:

$$\left(\frac{a+b}{2}\right)^p \leq 2\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)} \leq 2(a^p+b^p). \tag{4.1}$$

4.2. We remark that for the (nondecreasing) function considered in the example above our Theorem 2.1 (as expected) only give the fairly bad estimate

$$\left(\frac{a+b}{2}\right) \leq 4\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}.$$

This remark illustrates the advantage to work with the class $P(I)$ instead of $Q(I)$ in applications of Hadamard's inequality for monotone functions.

4.3. Theorems 2.1 and 3.1 can be generalized in various directions for example they can be formulated for the case when the functional A defined by

$$A(f) = \int_a^b f(x)dx$$

is replaced by a general isotonic linear functional A over a linear class of realvalued functions $g : T \to R$ (T is a nonempty set). The proof of this statement only consists of obvious modifications of our proofs.

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