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SOME CLASSES OF STARLIKE FUNCTIONS

BY

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Abstract. A well-known convolution operator $L(a,c)f = \phi(a,c) * f$, where f is analytic in the unit disc E and $\phi(a,c)$ is an incomplete beta function, is used to define the class $S^*(a,c)$. An analytic function $f \epsilon S^*(a,c)$ if $L(a,c)f$ is starlike in E. Coefficient result, a covering theorem and an inclusion relation is proved. It is also shown that this class is closed under convolution with convex function and an application of this result is given. We also discuss some radii problems for functions $f : f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \quad n = 1, 2, 3, \ldots$ which are starlike in E.

1. Introduction

Let $A(n)$ denote the class of functions f of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \quad n \epsilon N = \{1, 2, 3, \ldots\} \tag{1.1}$$

and analytic in the unit disk $E = \{z : | z | < 1\}$.

Let $P_n(\alpha)$ be the class of functions p of the form

$$p(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \cdots \tag{1.2}$$

which are analytic in E and satisfy $Rep(z) > \alpha$, for some $\alpha, 0 \leq \alpha < 1$ and for all $z \epsilon E$.

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A function $f \epsilon A(n)$ is said to be starlike of order α if and only if $\frac{zf'(z)}{f(z)} \epsilon P_n(\alpha)$ for some $\alpha (0 \leq \alpha < 1)$, and for all $z \epsilon E$. We denote this class as $S^*(n, \alpha)$. Similarly $f \epsilon A(n)$ is called a convex function of order α if and only if $\frac{(zf'(z))'}{f'(z)} \epsilon P_n(\alpha)$. We denote this class as $C(n, \alpha)$. Clearly $f \epsilon S^*(n, \alpha)$ if and only if $zf' \epsilon C(n, \alpha)$.

Let $A(1)$ be the class of analytic functions f defined on the unit disc $E = \{z : \mid z \mid < 1\}$, normalized by $f(0) = 0$ and $f'(0) = 1$. The class A is closed under the Hadamard product or convolution

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^{n+1},$$

where

$$f(z) = \sum_{n=0}^{\infty} a_n z^{n+1}, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n.$$

In particular, we consider convolution with the function $\phi(a, c)$ defined by

$$\phi(a, c, z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1}, \quad z \epsilon E, \qquad c \neq 0, -1, -2, \ldots,$$

where

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)},$$

i.e.

$$(a)_0 = 1, \quad (a)_n = a(a+1) \cdots (a+n-1), \quad n > 1.$$

The function $\phi(a, c)$ is an incomplete beta function, related to the Gauss hypergeometric function, defined by

$$\phi(a, c, z) = z_2 F_1(1, a; c, z).$$

It has an analytic continuation to the z-plane cut along the positive real line from 1 to ∞. We note that

$$\phi(a, 1, z) = \frac{z}{(1-z)^a}$$

and $\phi(2, 1, z)$ is the Koebe function. Carlson and Shaffer [3] defined a convolution operator on A involving an incomplete beta function as

$$L(a, c) f = \phi(a, c) * f, \quad f \epsilon A.$$

If $a = 0, -1, -2, \ldots$, then $L(a,c)f$ is a polynomial. If $a \neq 0, -1, -2, \ldots$, then application of the root test shows that the infinite series for $L(a,c)f$ has the same radius of convergence as that for f because

$$\lim_{n\to\infty} \left|\frac{(a)_n}{(c)_n}\right|^{\frac{1}{n}} = 1.$$

Hence $L(a,c)$ maps A into itself. The Ruscheweyh derivatives of f are $L(n+1,1)f$, $n = 0, 1, 2, \ldots$. $L(a,a)$ is the identity and if $a \neq 0, -1, -2, \ldots$, then $L(a,c)$ has a continuous inverse $L(c,a)$ and is a 1-1 mapping of A onto itself. $L(a,c)$ provides a convenient representation of differentiation and integration. If $g(z) = zf'(z)$, then $g = L(2,1)f$ and $f = L(1,2)g$.

We now define the following.

Definition 1.1. Let $f \epsilon A(1)$. Then, $f \epsilon S^*(a,c)$ if and only if $L(a,c)f \epsilon s^*(1,0)$ is starlike in E.

2. Preliminary Results

Lemma 2.1.([8]) *If $c \neq 0$, a and c are real and satisfy $a > N(c)$, where*

$$N(c) = \begin{cases} \mid c \mid + \frac{1}{2}, & \text{if } \mid c \mid \geq \frac{1}{3} \\ \frac{3c^2}{2} + \frac{2}{3}, & \text{if } \mid c \mid \leq \frac{1}{3} \end{cases} \tag{2.1}$$

then $\phi(c,a,z)$ is convex in E.

Lemma 2.2.([11]) *Let ψ be convex and g be starlike in E. Then, for F analytic in E with $F(0) = 1$, $\frac{\psi * Fg}{\psi * g}$ is contained in the convex hull of $F(E)$.*

Lemma 2.3.([7]) *Let u and v denote complex variables $u = u_1 + iu_2$, $v = v_1 + iv_2$ and let $\Phi(u,v)$ be a complex-valued function that satisfies the following conditions:*

(i) *$\Phi(u,v)$ is continuous in a domain $D \subset \mathbb{C}^2$.*

(ii) *$(1,0) \epsilon D$ and $\Phi(1,0) > 0$.*

(iii) *$Re\{\Phi(iu_2, v_1)\} \leq 0$ whenever $(iu_2, v_1) \epsilon D$ and $v_1 \leq -\frac{1}{2}(1 + u_2^2)$.*

If $P(z) = 1 + b_1z + b_2z^2 + \cdots$ is a function that is analytic in E such that $(p(z), zp'(z))\epsilon D$ and $Re\{\Phi(p(z), zp'(z))\} > 0$ holds for all $z\epsilon E$, then $Rep(z) > 0$ in E.

Lemma 2.4. *Let $p\ \epsilon P_n(0)$. Then, for $z\epsilon E$,*

(i)
$$\frac{1-r^n}{1+r^n} \leq Rep(z) \leq\mid p(z) \mid\leq \frac{1+r^n}{1-r^n}$$

(ii)
$$\mid zp'(z) \mid\leq \frac{2nr^n Rep(z)}{1-r^{2n}}$$

(iii)
$$\mid \frac{zp'(z)}{p(z)} \mid\leq \frac{2nr^n}{1-r^{2n}}.$$

For (i) we refer to [13], (ii) may be found in [2] and for (iii) see [6].

Lemma 2.5. *Let $g\epsilon S^*(n, 0)$. Then*

$$Re\frac{(zg'(z))'}{g'(z)} \geq \frac{1-2(n+1)r^n + r^{2n}}{1-r^{2n}}.$$

Proof. We have $zg'(z) = g(z)p(z)$, $p\ \epsilon P_n(0)$. So

$$\begin{aligned} Re\frac{(zg'(z))'}{g'(z)} &\geq Re\frac{zg'(z)}{g(z)} - \mid \frac{zp'(z)}{p(z)} \mid \\ &\geq \frac{1-r^n}{1+r^n} - \frac{2nr^n}{1-r^{2n}} = \frac{1-2(n+1)r^n + r^{2n}}{1-r^{2n}}, \end{aligned}$$

where we have used Lemma 2.4.

3. The Class $S^*(a, c)$

Theorem 3.1. *Let $f\epsilon S^*(a, c)$ and be given by*

$$f(z) = \sum_{n=0}^{\infty} a_{n+1}z^{n+1} (a_1 = 1).$$

Then, for $n \geq 1$,

$$\mid a_{n+1} \mid\leq \frac{(n+1)(c)_n}{(a)_n}.$$

This result is sharp with equality for the function f_0, where

$$L(a,c)f_0 = \frac{z}{(1-z)^2}.$$

The proof follows immediately from the well known coefficient result for the class S^* of starlike functions and the fact that $f \epsilon S^*(a,c)$.

Theorem 3.2. *Let $f \epsilon S^*(a,c)$, where a and c satisfy the conditions of Lemma 2.1. Then, for $z \epsilon E$, f is starlike and hence univalent.*

Proof. Since $L(a,c) f \epsilon S^*$, we have

$$Re\left[\frac{z(\phi(a,c)*f)'}{\phi(a,c)*f}\right] = Re\left[\frac{\phi(a,c)*zf'}{\phi(a,c)*f}\right] > 0, \quad z \epsilon E.$$

Now

$$\begin{aligned}\frac{zf'(z)}{f(z)} &= \frac{\phi(a,c,z)*zf'(z)*\phi(c,a,z)}{\phi(a,c,z)*f(z)*\phi(c,a,z)} \\ &= \frac{\phi(c,a,z)*\frac{\phi(a,c,z)*zf'(z)}{\phi(a,c,z)*f(z)}(\phi(a,c,z)*f(z))}{\phi(c,a,z)*(\phi(a,c,z)*f(z))}.\end{aligned}$$

Since $\phi(c,a,z)$ is convex by Lemma 2.1 and $f \epsilon S^*(a,c)$, we can use Lemma 2.2 to conclude that f is starlike.

Theorem 3.3. (Covering Theorem). *Let $f \epsilon S^*(a,c)$ with a and c satisfying* (2.1). *Then the disc E is mapped onto a domain that contains the disc*

$$|\ \omega\ | < \frac{2(c+a)}{a}.$$

Proof. Let ω_0 be any complex number such that $f(z) \neq \omega_0$ for $z \epsilon E$. Then

$$\frac{\omega_0 f(z)}{\omega_0 - f(z)} = z + (a_2 + \frac{1}{\omega_0})z^2 + \cdots$$

is univalent by Theorem 3.2, and hence

$$\left| a_2 + \frac{1}{\omega_0} \right| \leq 2.$$

Now using Theorem 3.1 for $n = 2$, we obtain the required result.

We now prove an inclusion result.

Theorem 3.4. *Let $a \neq 0$, c and a be real and $c > N(d)$, where $N(d)$ is defined as in* (2.1)*. Then*

$$S^*(a,d) \subset S^*(a,c).$$

Proof. Let $f \epsilon S^*(a,d)$. Then

$$\begin{aligned} \frac{z(\phi(a,c)*f)'}{\phi(a,c)*f} &= \frac{\phi(a,c)*zf'}{\phi(a,c)*f} \\ &= \frac{(\phi(a,d)*\phi(d,c))*zf'}{(\phi(a,d)*\phi(d,c))*f} \\ &= \frac{(\phi(d,c)*\left(\frac{\phi(a,d)*zf'}{\phi(a,d)*f}\right)(\phi(a,d)*f)}{\phi(d,c)*(\phi(a,d)*f)}. \end{aligned}$$

Since, by Lemma 2.1, $\phi(d,c)$ is convex and $f \epsilon S^*(a,d)$, we apply Lemma 2.2 to have the required result that $f \epsilon S^*(a,c)$.

Theorem 3.5. *Let $a \neq 0$ and c be real and satisfy $c > N(a)$, where $N(a)$ is defined in the similar way of* (2.1)*. Let ψ be a convex function in E. If $f \epsilon S^*(a,c)$ then $\psi * f \epsilon S^*(a,c)$.*

Proof. Since

$$\frac{z[\phi(a,c)*(\psi*f)]'}{\phi(a,c)*(\psi*f)} = \frac{\psi*\left(\frac{\phi(a,c)*zf'}{\phi(a,c)*f}\right)(\phi(a,c)*f)}{\psi*(\phi(a,c)*f)},$$

the result follows immediately by using Lemma 2.2. This shows that the class $S^*(a,c)$ is closed under convolution with convex functions.

The following is an application of Theorem 3.5.

Theorem 3.6. *Let $f \epsilon S^*(a,c)$ with a and c satisfying the conditions of Theorem* 3.5*. For $0 < \lambda < 1$, let*

$$F(z) = (1-\lambda)f(z) + \lambda z f'(z). \tag{3.1}$$

Then $F\epsilon S^*(a,c)$ *for* $\mid z \mid < r_0$, *where*

$$r_0 = 1/[2\lambda + \sqrt{4\lambda^2 - 2\lambda + 1}]. \tag{3.2}$$

On the other hand, if $F\epsilon S^*(a,c)$, *then* $f\epsilon S^*(a,c)$ *for* $z\epsilon E$.

Proof. We can write (3.1) as

$$F(z) = (\psi * f)(z),$$

where

$$\begin{aligned}\psi(z) =&(1-\lambda)\frac{z}{1-z} + \lambda\frac{z}{(1-z)^2}, \quad 0 < \lambda < 1\\ =&z + \sum_{n=2}^{\infty}[1 + (n-1)\lambda]z^n.\end{aligned}$$

The function ψ is convex for $\mid z \mid < r_0$ where r_0 is given by (3.2) and this radius is best possible. Thus, applying Theorem 3.5, we see that $F\epsilon S^*(a,c)$ for $\mid z \mid < r_0$.

Also, from (3.1), we have

$$f(z) = \left(\sum_{n=1}^{\infty}\frac{1}{1+(n-1)\lambda}z^n\right) * F(z).$$

Since

$$\sum_{n=1}^{\infty}\frac{1}{1+(n-1)\lambda}z^n$$

is convex, see [10], we obtain the required result that $f\epsilon S^*(a,c)$.

For a function $f\epsilon A$, we define the integral operator I_β by

$$I_\beta(f) = \frac{\beta+1}{z^\beta}\int_0^z t^{\beta-1}f(t)dt. \quad (\beta > -1) \tag{3.3}$$

The operator I_β, when $\beta\epsilon N = \{1,2,3,\ldots\}$ was introduced by Bernardi [1]. In particular I_1 was studied earlier by Libera [4] and Livingston [5].

We prove the following.

Theorem 3.7. *Let* $f\epsilon S^*(a,c)$ *and let* $I_\beta(f)$ *be defined by* (3.3) *with* $\beta\epsilon N$. *Then, for* $z\epsilon E$,

$$Re\left\{\frac{z[L(a,c)I_\beta(f)]'}{L(a,c)I_\beta(f)}\right\} > \alpha,$$

where

$$\alpha = \frac{-(2\beta+1)+\sqrt{4\beta^2+4\beta+9}}{4}. \tag{3.4}$$

Proof. Let

$$\frac{z[L(a,c)I_\beta(f)]'}{L(a,c)I_\beta(f)} = (1-\alpha)p(z)+\alpha, \tag{3.5}$$

with α given by (3.4). We see that p is analytic in E and $p(0)=1$. Differentiating both sides of (3.5) logarithmically we have

$$\left\{\frac{z^2[L(a,c)I_\beta(f)]''+z[L(a,c)I_\beta(f)]'}{z[L(a,c)I_\beta(f)]'} - \frac{z[L(a,c)I_\beta(f)]'}{L(a,c)I_\beta(f)}\right\} = \frac{(1-\alpha)zp'(z)}{(1-\alpha)p(z)+\alpha}. \tag{3.6}$$

We note that

$$z[L(a,c)I_\beta(f)]' = (\beta+1)L(a,c)f(z) - \beta L(a,c)I_\beta(f). \tag{3.7}$$

From this it follows that

$$z^2[L(a,c)I_\beta(f)]'' = (\beta+1)z[L(a,c)f(z)]' - \beta L(a,c)I_\beta(f). \tag{3.8}$$

Substituting (3.7) and (3.8) into (3.6) and simplifying we obtain

$$\frac{z[L(a,c)f(z)]'}{L(a,c)f(z)} = (1-\alpha)p(z)+\alpha+\frac{(1-\alpha)zp'(z)}{(1-\alpha)p(z)+\alpha+\beta}.$$

Since $f\epsilon S^*(a,c)$, we have

$$Re\left[(1-\alpha)p(z)+\alpha+\frac{(1-\alpha)zp'(z)}{(1-\alpha)p(z)+\alpha+\beta}\right] > 0, \qquad z\epsilon E.$$

We can now form the functional $\Phi(u,v)$ by choosing $u=p(z), v=zp'(z)$. Thus

$$\Phi(u,v) = (1-\alpha)u+\alpha+\frac{(1-\alpha)v}{(1-\alpha)u+(\alpha+\beta)}.$$

The first two conditions of Lemma 2.3 are clearly satisfied. We proceed to verify the condition (iii) as follows.

$$Re[\Phi(iu_2,v_1)] = \alpha+\frac{(1-\alpha)(\alpha+\beta)v_1}{(\alpha+\beta)^2+(1-\alpha)^2u_2^2}.$$

By putting $v_1 \le \frac{-(1+u_2^2)}{2}$. we obtain

$$\begin{aligned}
Re[\Phi(iu_2, v_1)] &\le \alpha - \frac{1}{2}\frac{(1-\alpha)(\alpha+\beta)(1+u_2^2)}{(\alpha+\beta)^2+(1-\alpha)^2u_2^2}\\
=&\frac{2\alpha(\alpha+\beta)^2+2\alpha(1-\alpha)^2u_2^2-(1-\alpha)(\alpha+\beta)-(1-\alpha)(\alpha+\beta)u_2^2}{2[(\alpha+\beta)^2+(1-\alpha)^2u_2^2]}\\
=&\frac{A+Bu_2^2}{2C},
\end{aligned}$$

where

$$\begin{aligned}
A =&2\alpha(\alpha+\beta)^2-(1-\alpha)(\alpha+\beta),\\
B =&2\alpha(1-\alpha)^2-(1-\alpha)(\alpha+\beta),\\
C =&(\alpha+\beta^2)+(1-\alpha)^2u_2^2>0.
\end{aligned}$$

We notice that $Re[\Phi(iu_2, v_1)] \le 0$ if and only if $A \le 0$ and $B \le 0$ which means $Rep(z) > 0$ in E.

From $A \le 0$, we obtain α as defined by (3.4) and $B \le 0$ implies that $0 < \alpha < 1$.

We note that, for $\beta = 1$, we have

$$\alpha = \frac{-3+\sqrt{17}}{4}.$$

4. Some Radii Problems

Theorem 4.1. *Let $F \epsilon S^*(n,\alpha)$, $0 \le \alpha < 1$ and let f, for $\lambda \ne \frac{1}{2(1-\alpha)}$, be defined as*

$$f(z) = (1-\lambda)F(z) + \lambda zF'(z), \quad z\epsilon E. \tag{4.1}$$

Then $f\epsilon S^*(n,\alpha)$ for $\mid z \mid < r_0$, where

$$\left.\begin{aligned}
r_0 \quad &= \left\{\frac{A+\sqrt{A^2+B}}{B}\right\}^{\frac{1}{n}},\\
A \quad &= \lambda(\alpha-n-1), B = 1-2\lambda+2\alpha\lambda
\end{aligned}\right\}. \tag{4.2}$$

This result is best possible.

Proof. We can write (4.1) as

$$f(z) = \lambda z^{2-\frac{1}{\lambda}}(z^{\frac{1}{\lambda}-1}F(z))',$$

and we obtain

$$F(z)=\frac{1}{\lambda}z^{1-\frac{1}{\lambda}}\int_0^z z^{\frac{1}{\lambda}-2}f(z)dz. \tag{4.3}$$

Now $F\epsilon S^*(n,\alpha)$ implies that $\frac{zF'(z)}{F(z)}=p(z)$, $p\epsilon P_n(\alpha)$. So, from (4.3), we have

$$p(z)\int_0^z z^{\frac{1}{\lambda}-2}f(z)dz=(1-\frac{1}{\lambda})\int_0^z z^{\frac{1}{\lambda}-2}f(z)dz+z^{\frac{1}{\lambda}-1}f(z). \tag{4.4}$$

Differentiating both sides of (4.4) and simplifying, we have

$$\frac{zf'(z)}{f(z)}=p(z)+\left(p'(z)\frac{\int_0^z z^{\frac{1}{\lambda}-2}f(z)dz}{z^{\frac{1}{\lambda}-2}f(z)}\right). \tag{4.5}$$

Since $p\epsilon P_n(\alpha)$, there exists $h\epsilon P_n(0)$ such that

$$p(z)=(1-\alpha)h(z)+\alpha. \tag{4.6}$$

Using (4.6), (4.5) can be written as

$$\frac{1}{(1-\alpha)}\left\{\frac{zf'(z)}{f(z)}-\alpha\right\}=h(z)+\left(zh'(z)\frac{\int_0^z z^{\frac{1}{\lambda}-2}f(z)dz}{z^{\frac{1}{\lambda}-1}f(z)}\right). \tag{4.7}$$

Now

$$\begin{aligned}\frac{z^{\frac{1}{\lambda}-1}f(z)}{\int_0^z z^{\frac{1}{\lambda-2}}f(z)dz}=&\frac{\lambda z(z^{\frac{1}{\lambda}-1}F(z))'}{\lambda(z^{\frac{1}{\lambda}-1}F(z))}\\=&\frac{zF'(z)}{F(z)}+\frac{1}{\lambda}-1\\=&(1-\alpha)h(z)+\alpha+\frac{1}{\lambda}-1.\end{aligned} \tag{4.8}$$

Hence, using Lemma 2.4, we obtain from (4.7) and (4.8),

$$\begin{aligned}\frac{1}{(1-\alpha)}Re\left\{\frac{zf'(z)}{f(z)}-\alpha\right\}\geq&Reh(z)\left[1-\frac{2nr^n}{(1-r^{2n})}\frac{(1+r^n)}{(\frac{1}{\lambda}+\left\{\frac{1}{\lambda}-2(1-\alpha)\right\}r^n)}\right]\\=&Reh(z)\left[\frac{(1-r^n)\{1+(1-2\lambda+2\alpha\lambda)r^n\}-2n\lambda r^n}{(1-r^n)\{1+(1-2\lambda+2\alpha\lambda)r^n\}}\right].\end{aligned}$$

The right hand side is positive for $|z|<r_0$, where r_0 is given by (4.2) and this gives us the required result. This result is best possible as can be seen from the function

$$f_{\alpha,\lambda}(z)=(1-\lambda)F_\alpha(z)+\lambda zF'_\alpha(z),$$

where

$$F_\alpha(z) = z(1-z^n)^{\frac{(2\alpha-2)}{n}} \epsilon S^*(n,\alpha).$$

Special Cases

(i) Let $\lambda = \frac{1}{2}$, $0 < \alpha < 1, F\epsilon S^*(1,\alpha)$. Then $f\epsilon S^*(1,\alpha)$ for $|z| < r = \{(\alpha + 2) + (\alpha^2+4)^{\frac{1}{2}}\}/2\alpha$. This result is proved in [9] for $0 < \alpha \le 1/2$.

(ii) With $\alpha = 0$, $\lambda \neq \frac{1}{2}$, we have $f\epsilon S^*(n,0)$ for $|z| < r = 1/\{(n+1)\lambda + \sqrt{(n+1)^2\lambda^2 + (1-2\lambda)}\}$.

Using the same technique, we can prove the following results.

Theorem 4.2. *Let $F\epsilon C(n,\alpha)$, $0 \le \alpha < 1$, $n\epsilon N$ and let, for $\lambda \neq \frac{1}{2(1-\alpha)}$, f be defined by* (4.1). *Then $f\epsilon C(n,\alpha)$ for $\mid z \mid < r_0$ where r_0 is given by* (4.2). *This result is sharp.*

Theorem 4.3. *Let $g\epsilon S^*(n,\alpha)$ and $\frac{zF'}{G}\epsilon P_n(\alpha)$ for $z\epsilon E$. Let, for $\lambda \neq \frac{1}{2(1-\alpha)}$, f be defined by* (4.1) *and let*

$$g(z) = (1-\lambda)G(z) + \lambda z G'(z). \tag{4.9}$$

Then $\frac{zf'}{g}\epsilon P_n(\alpha)$ for $\mid z \mid < r_0$, where r_0 is given by (4.2).

This result is best possible. This can be seen by taking

$$F_\alpha(z) = G_\alpha(z) = z(1-z)^{(2\alpha-2)/n}.$$

We now prove the following.

Theorem 4.4. *Let $G\epsilon S^*(n,0)$ and $\frac{(zF')'}{G'}\epsilon P_n(\alpha)$ for $0 \le \alpha < 1$ and $z\epsilon E$. Let, for $\lambda > 0, \lambda \neq 1/2$, f be defined by* (4.1) *and g be defined by* (4.9). *Then $\frac{(zf')'}{g'}\epsilon P_n(\alpha)$ for $\mid z \mid < r_2$, where $r_2 = \min(r_0, r_1)$, where r_0 is given by* (4.2) *with $\alpha = 0$ and*

$$r_1 = \left\{\lambda(2n+1) + \sqrt{\lambda^2(2n+1)^2 - 2\lambda + 1}\right\}^{\frac{-1}{n}}. \tag{4.10}$$

Proof. We can write (4.1) as

$$F(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z z^{\frac{1}{\lambda}-2} f(z) dz.$$

So

$$\begin{aligned}\frac{(zF'(z))'}{G'(z)} =& \frac{z^{\frac{1}{\lambda}} f'(z) - (\frac{1}{\lambda} - 1) \int_0^z z^{\frac{1}{\lambda}-1} f'(z) dz}{\int_0^z z^{\frac{1}{\lambda}-1} g'(z) dz} \\ =&(1-\alpha)h(z) + \alpha, \qquad h\epsilon P_n(0), z\epsilon E.\end{aligned}$$

Differentiating and simplifying, we obtain

$$Re\left[\frac{(zf'(z))'}{g'(z)} - \alpha\right] \geq (1-\alpha)Reh(z)\left[1 - \frac{2nr^n}{1-r^{2n}} \mid \frac{\int_0^z z^{\frac{1}{\lambda}-1} g'(z) d(z)}{z^{\frac{1}{z}} g'(z)} \mid\right], \tag{4.11}$$

where we have used Lemma 2.4. Now

$$\begin{aligned}\frac{z^{\frac{1}{\lambda}} g'(z)}{\int_0^z z^{\frac{1}{\lambda}-1} g'(z) dz} =& \frac{\frac{1}{\lambda} G'(z) + zG''(z)}{G'(z)} \\ =&(\frac{1}{\lambda} - 1) + \frac{(zG'(z))'}{G'(z)}.\end{aligned} \tag{4.12}$$

From (4.11), (4.12) and Lemma 2.5, we have

$$\begin{aligned}&\frac{1}{(1-\alpha)} Re\left[\frac{(zf'(z))'}{g'(z)} - \alpha\right] \\ \geq& Reh(z)\left[1 - \frac{2nr^n}{(1-r^{2n})} \frac{(1-r^{2n})}{\{\frac{1}{\lambda} - 2(n+1)r^n + (2-\frac{1}{\lambda})r^{2n}\}}\right] \\ =& Reh(z)\left[\frac{1 - 2\lambda(2n+1)r^n + (2\lambda-1)r^{2n}}{1 - 2\lambda(n+1)r^n + (2\lambda-1)r^{2n}}\right].\end{aligned}$$

The right hand side is positive for $\mid z \mid < r_1$, where r_1 is given by (4.10). We note that, for $n = 1$, $\lambda \neq \frac{1}{2}$,

$$r_1 = 1/\{3\lambda + \sqrt{9\lambda^2 - 2\lambda + 1}\}.$$

The restriction on λ in Theorem 4.1 can be removed for the case $n = 1$, and we prove the result as follows.

Theorem 4.5. *Let* $F \epsilon S^*(1,\alpha)$, $0 \le \alpha < 1$ *and for* $\lambda > 0$, *let* f *be defined by* (4.1). *Then* $f \epsilon S^*(1,\alpha)$ *for* $\mid z \mid < R$ *where*

$$R = \frac{\mid \mu + 1 \mid}{[A + (A^2 - \mid \mu^2 - 1 \mid^2)^{\frac{1}{2}}]^{\frac{1}{2}}}, \tag{4.13}$$

$$\mu = \frac{\alpha\lambda - \lambda + 1}{\lambda(1-\alpha)}, \qquad s = \frac{1}{(1-\alpha)}$$

and

$$A = \{2(s+1)^2 + \mid \mu \mid^2 - 1\}.$$

This result is best possible.

Proof. With $\frac{zF'(z)}{F(z)} = (1-\alpha)h(z) + \alpha$, $h \epsilon P_1(0)$, we proceed on the same lines as before and have

$$\frac{1}{(1-\alpha)} Re\left[\frac{zf'(z)}{f(z)} - \alpha\right] = h(z) + \frac{zh'(z)}{(1-\alpha)h(z) + (\alpha + \frac{1}{\lambda} - 1)}$$

$$= h(z) + \frac{(\frac{1}{1-\alpha})zh'(z)}{\frac{\alpha\lambda-\lambda+1}{\lambda(1-\alpha)} + h(z)}.$$

Now Ruscheweyh and Singh [12] have shown that, for $h \epsilon P_1(0), s > 0, \mu \ne -1$ (complex),

$$Re\left[h(z) - \frac{szh'(z)}{h(z) + \mu}\right] > 0,$$

for $\mid z \mid < R$, where R is given by (4.13) and this result is best possible. Taking $s = \frac{1}{1-\alpha}$ and $\mu = \frac{\alpha\lambda-\lambda+1}{\lambda(1-\alpha)}$, we obtained the required result.

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