SOOCHOW JOURNAL OF MATHEMATICS
Volume 24, No. 4, pp. 255-259, October 1998

PRODUCT SUBMANIFOLDS OF ALMOST r-PARACONTACT RIEMANNIAN MANIFOLDS OF P-SASAKIAN TYPE

BY

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Abstract. In this paper it is shown that for an almost r-paracontact manifold of P-Sasakian type there exists a product submanifold.

1. Introduction

By studying properties of integral submanifolds of a certain integrable distribution on an almost r-paracontact Riemannian manifold of P-Sasakian type ([2]), it is shown that there exists a product submanifold of this manifold.

2. Almost r-Paracontact Riemannian Manifolds

The definitions of almost r-paracontact Riemannian ([1]) and some types of these manifolds ([2]) together with some of their properties are presented in this section.

Definition 2.1. Let M_n be an n-dimensional Riemannian manifold with a positive definite metric g. If on M_n there exist: a tensor field ϕ of type (1,1), r vector fields $\xi_1, \xi_2, \ldots, \xi_r$ (r<n), r 1-forms $\eta^1, \eta^2, \ldots, \eta^r$ such that

$$\eta^\alpha(\xi_\beta) = \delta^\alpha_\beta, \qquad \alpha, \beta \in (r) = 1, 2, \ldots, r, \tag{2.1}$$

$$\phi^2 = Id - \eta^\alpha \otimes \xi_\alpha, \tag{2.2}$$

$$\eta^\alpha(X) = g(X, \xi_\alpha), \qquad \alpha \in (r), \tag{2.3}$$

Received September 24, 1997; revised November 25, 1997.
AMS Subject Classification. 53C40, 53C15.
Key words. Riemannian product submanifolds, almost r-paracontact structures.

$$g(\phi X,\phi Y)=g(X,Y)-\sum_\alpha \eta^\alpha(X)\eta^\alpha(Y), \tag{2.4}$$

where X, Y are vector fields on M_n and $a^\alpha b_\alpha \overset{\text{def}}{=} \sum_\alpha a^\alpha b_\alpha$, then $\Sigma=(\phi,\xi_\alpha,\eta^\alpha, g)_{\alpha\in(r)}$ is said to be an *almost r-paracontact Riemannian structure* on M_n, and M_n is an *almost r-paracontact Riemannian manifold.*

Remark 2.1. From (2.1) through (2.4) it follows that

$$\phi(\xi_\alpha)=0, \qquad \alpha\in(r), \tag{2.5}$$

$$\eta^\alpha\circ\phi=0, \qquad \alpha\in(r), \tag{2.6}$$

$$\Phi(X,Y)\overset{\text{def}}{=}g(\phi X,Y)=g(X,\phi Y). \tag{2.7}$$

From (2.7) we get

$$\nabla_Z\Phi(X,Y)=g((\nabla_Z\phi)X,Y). \tag{2.8}$$

Definition 2.2. If M_n is an almost r-paracontact manifold with the structure $\Sigma=(\phi,\xi_\alpha,\eta^\alpha,g)_{\alpha\in(r)}$, then Σ is said to be normal if an almost product structure F defined on $M_n\times R^r$ by $F(X,f^\alpha\frac{d}{dt^\alpha})=(\phi X+f^\alpha\xi_\alpha,\eta^\alpha(X)\frac{d}{dt^\alpha})$ is integrable, i.e., its Nijenhuis tensor field N_F vanishes.

Theorem 2.1. *An almost r-paracontact structure Σ on M_n is normal if and only if $N(X,Y)=N_\phi(X,Y)-2d\eta^\alpha(X,Y)\xi_\alpha=0$, where N_ϕ is the Nijenhuis tensor field for ϕ.*

Definition 2.3. An almost r-paracontact Riemannian manifold M_n with a structure $\Sigma=(\phi,\xi_\alpha,\eta^\alpha,g)_{\alpha\in(r)}$ is said to be of *P-Sasakian type* if

$$\Phi(X,Y)=(\nabla_Y\eta^\alpha)(X), \quad \text{for all } \alpha\in(r), \tag{2.9}$$

$$(\nabla_Z\Phi)(X,Y)=-\sum_\alpha \eta^\alpha(X)g(\phi Y,\phi Z)-\sum_\alpha \eta^\alpha(Y)g(\phi X,\phi Z). \tag{2.10}$$

Theorem 2.2. *An almost r-paracontact Riemannian structure $\Sigma=(\phi,\xi_\alpha, \eta^\alpha,g)_{\alpha\in(r)}$ of P-Sasakian type is normal and all η^α are closed.*

Remark 2.2. The conditions (2.9) and (2.10) are equivalent to:

$$\phi X=\nabla_X\xi_\alpha, \qquad \text{for all } \alpha\in(r), \tag{2.11}$$

$$(\nabla_Y\phi)(X) = -\sum_\beta \eta^\beta(X)[Y-\eta^\alpha(Y)\xi_\alpha]$$
$$-[g(X,Y)-\sum_\alpha \eta^\alpha(X)\eta^\alpha(Y)]\sum_\beta \xi_\beta. \tag{2.12}$$

3. Product Submanifolds of Almost r-Paracontact Riemannian Manifolds of P-Sasakian type

In this section properties of integral sumbanifolds of certain integrable distributions on an almost r-paracontact Riemannian manifold of P-Sasakian type are studied.

Let M_n be an almost r-paracontact Riemannian manifold with the structure $\Sigma = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$. The tensor ϕ has constant eigenvalues 1, -1, and 0. Let p, q, and r be their multiplicities, respectively. Then M_n is said to be of type (p,q). If T_xM_n is the tangent space to M_n at a point $x\in M_n$, then M_n has an orthonormal frame $\{e_1,\dots,e_p,e_{p+1},\dots,e_{p+\lambda},e_{p+q+1}=\xi_1,\dots,e_n=\xi_r\}$ such that $\phi(e_a)=e_a$, $\phi(e_{p+\lambda})=-e_{p+\lambda}$, $\phi(e_{p+q+\alpha})=0$, $a=1,\dots,p,\lambda=1,\dots,q,\alpha=1,\dots,r$.

We can define the following distributions on an almost r-paracontact manifold M_n with the structure $\Sigma = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$ ([2]):

$$\begin{aligned}
D^+ &= \{X;\ \phi X = X\}, && \dim D^+ = \text{p}, \\
D^- &= \{X;\ \phi X = -X\}, && \dim D^- = \text{q}, \\
D^0 &= \{X;\ \phi X = 0\}, && \dim D^0 = \text{r}, \qquad (\text{p+q+r=n}) \\
D &= \{X;\ \eta^1(X) = \cdots = \eta^r(X) = 0\}.
\end{aligned} \tag{3.1}$$

Remark 3.1. $D = D^+ \oplus D^-$ and $TM_n = D \oplus D^0$.

We also have:

Theorem 3.1.([1]) *An almost r-paracontact structure Σ on M_n is normal if and only if $\mathcal{L}_{\xi_\alpha}\eta^\beta = 0$, $[\xi_\alpha,\xi_\beta]=0$, $\alpha,\beta=1,2,\dots,r$, and the distributions $D^+, D^-, D^+\oplus D^0$, $D^-\oplus D^0$ are integrable, where $\mathcal{L}_X$ is the Lie differentiation operator with respect to a vector field X.*

Let (M_n,Σ) be an almost r-parcaontact Riemannian manifold of P-Sasakian type. Then we have:

Theorem 3.2. *An almost r-paracontact Riemannian manifold M_n of P-Sasakian type with the structure $\Sigma = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$ of type* (p,q) *with p≠q is never compact.*

Proof. From (2.11), for any ξ_α we have $\text{div}\xi_\alpha = \text{trace}\{X \to \nabla_X\xi_\alpha\} = \sum_{i=1}^{n} g(\nabla_{e_i}\xi_\alpha, e_i) = \sum_{i=1}^{n} g(\phi e_i, e_i) = \text{p} - \text{q}$. Suppose that M_n is compact and orientable. Then by Green's theorem we get $0 = \int_{M_n}\text{div}\xi_\alpha \, dM_n = (\text{p}-\text{q})\int_{M_n} dM_n \neq 0$. Hence, M_n cannot be compact and orientable. Now, assume that M_n is compact but not orientable. Then a double covering manifold $\tilde{M}_n$ of M_n is compact and orientable and M_n induces an almost r-paracontact structure of the same type on $\tilde{M}_n$, so again we come to a contradiction.

Remark 3.2. The case p=q is still an open problem.

Proposition 3.1. *On an almost r-paracontact Riemannian manifold M_n of P-Sasakian type with a structure Σ, the distributions D^+, D^-, D^0, D given by* (3.1) *are integrable.*

Proof. From Corollary 2.1 all η^α are closed and Σ is normal. Thus, from Theorem 3.1 the distributions D^+, D^- and D^0 are integrable. Now, let $X, Y \in D$. Then we have $0 = 2d\eta^\alpha(X,Y) = X\eta^\alpha(Y) - Y\eta^\alpha(X) - \eta^\alpha[X,Y]$. Hence $\eta^\alpha[X,Y] = 0$ and D is integrable.

Proposition 3.2. *The trajectories of each structure vector field ξ_α on M_n are geodesics.*

Proof. From (2.1) and (2.3) we have $1 = g(\xi_\alpha, \xi_\alpha)$ and $g(\nabla_X\xi_\alpha, \xi_\alpha) = 0$. From (2.3) we get $(\nabla_X\eta^\alpha)Y = g(\nabla_X\xi_\alpha, Y)$. Since η^α are closed, we get $(\nabla_X\eta^\alpha)Y = (\nabla_Y\eta^\alpha)X$. Hence, we obtain $g(\nabla_X\xi_\alpha, \xi_\alpha) = (\nabla_X\eta^\alpha)\xi_\alpha = (\nabla_{\xi_\alpha}\eta^\alpha)X$ $= g(\nabla_{\xi_\alpha}\xi_\alpha, X) = 0$. Thus, $\nabla_{\xi_\alpha}\xi_\alpha = 0$.

Definition 3.1. Let $\bar{D}$ be a distribution on an almost r-paracontact Riemannian manifold (M_n, Σ). If, for any vector field $X \in \bar{D}$ and any vector field Y, $\nabla_Y X$ is in $\bar{D}$, then $\bar{D}$ is said to be parallel.

Theorem 3.3. *If M_n is an almost r-paracontact Riemannian manifold of P-Sasakian type with a structure $\Sigma = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$, and if M is a maximal* *integral submanifold of the distribution D given by* (3.1) *on M_n, then M is locally a Riemannian direct product $M^+ \times M^-$, where M^+ and M^- are maximal integral submanifolds of the distributions D^+ and D^- given by* (3.1), *respectively.*

Proof. Since, from Remark 3.1 and Proposition 3.1, D is an integrable direct sum of the integrable distributions D^+ and D^-, it suffices to show that D^+ and D^- are parallel. For $X \in D$ we have $\eta^\alpha(X) = 0$, and $\phi X = \bar{\phi}X + \bar{F}X$, where $\bar{\phi}X \in D$ and $\bar{F}X \in D^0$. Making use of (2.6) and $(3.1)_4$ we obtain $\eta^\alpha \bar{F}X = 0$, $\alpha \in (r)$, which means that $\bar{F}X = 0$; hence $\phi X = \bar{\phi}X$ and $\bar{\phi}^2 = Id_M$, which means that $\bar{\phi}$ is an almost product structure on M. Let h be the second fundamental tensor of M. Then the Gauss formula is given by $\nabla_Y X = \bar{\nabla}_Y X + h(X,Y)$, where $h(X,Y) \in D^0$ and $\bar{\nabla}$ denotes the operator of covariant differentiation on M. Since $h(X,Y) \in D^0$, then for $X, Y \in D$ we have $\phi h(X,Y) = 0$. Now, we show that D^+ is parallel, that is $\bar{\nabla}_Y X \in D^+$ for any $X \in D^+$ and $Y \in D$. We have $\bar{\phi}(\bar{\nabla}_Y X) = \phi(\bar{\nabla}_Y X) = \phi(\nabla_Y X) = \phi(\nabla_Y(\phi X)) = (\phi \nabla_Y \phi)X + \phi^2(\nabla_Y X)$, or $\bar{\phi}(\bar{\nabla}_Y X) = (\phi \nabla_Y \phi)X + \bar{\nabla}_Y X$. Making use of (2.12) and (2.5) we obtain $(\phi \nabla_Y \phi)X = -\phi(g(X,Y)\sum_\beta \xi_\beta) = 0$. Hence we get $\bar{\phi}(\bar{\nabla}_Y X) = \bar{\nabla}_Y X$ which means that D^+ is parallel. In similar way we show that D^- is also parallel.

Corollary 3.1. *If M_n is an almost r-paracontact Riemannian manifold of P-Sasakian type with a structure $\Sigma = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha \in (r)}$, then there exists a submanifold M of M_n that is locally a Riemannian product manifold.*

References

[1] A. Bucki, A. Miernowski, *Almost r-paracontact structures*, Ann. Univ. Mariae Curie-Sklodowska, Sect. A, 39 (1985), 13-26.

[2] A. Bucki, *Almost r-paracontact structures of P-Sasakian type*, Tensor, N. S., 42 (1985), 42-54.

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