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ON THE AREA OF NUMERICAL RANGE

BY

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Abstract. Let A be an n-by-n complex matrix. The numerical range of A is the set $W(A) = \{x^*Ax : x \in \mathbf{C}^n, x^*x = 1\}$. We discuss several methods to find upper bounds for the area of W(A) for some 3-by-3 upper triangular nilpotent matrices. The area of the numerical range of special matrices is treated for general dimension.

1. Introduction

Let $A \in M_n(\mathbf{C})$, the set of all n-by-n complex matrices. The numerical range of A is the set of complex numbers

$$W(A) = \{x^*Ax : x^*x = 1, x \in \mathbf{C}^n\}.$$

By the well-known Toeplitz-Hausdorff theorem, the numerical range W(A) is a convex subset of $\mathbf{R}^2$. The numerical range of a matrix has been well studied, see [5] for further results and references.

The area of the numerical range is an interesting topics in the study of numerical range. We use enveloping method, Newton's method and Geršgorin disc theorem, to find upper bounds for the area of the numerical range of upper triangular nilpotent matrices and some special matrices. The numerical range of a 3-by-3 matrix is in general an oval [7]. Marcus and Pesce [9] characterized the numerical range of a 3-by-3 matrix to be a circular disc. In section 2, we use

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enveloping and Geršgorin disc theorem to find an upper bound for the area of the numerical range of the following 3-by-3 upper triangular nilpotent matrix

$$A = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \in M_3(\mathbf{C}). \tag{1}$$

In section 3, Newton's method (cf.[3]) for approximating the maximal root of the characteristic polynomial of $H_\theta = (e^{i\theta}A + e^{-i\theta}A^*)/2$ is applied to find an upper bound for the area of the numerical range of the matrix (1). In section 4, we use a result of Johnson [6] to find an upper bound for the area of the numerical range of an upper triangular nilpotent matrix. In section 5, we give parametrization of the numerical ranges of idempotent matrices and certain form of matrices, the area of the numerical range of these classes of matrices is computed.

2. Geršgorin and Envelope

Let $A = (a_{ij}) \in M_n(\mathbf{C})$, for $1 \leq i \leq n$, denote

$$R_i(A) = \sum_{j=1, j\neq i}^{n} |a_{ij}|;$$

$$C_i(A) = \sum_{j=1, j\neq i}^{n} |a_{ji}|.$$

Then Geršgorin disc theorem(cf.[4]) assures all eigenvalues of A are contained in the union of n discs

$$\bigcup_{i=1}^{n} \{z \in \mathbf{C} : |z - a_{ii}| \leq R_i(A)\} = G(A).$$

Since A and A^T have the same eigenvalues, one may obtain a Geršgorin disc theorem by applying the Geršgorin disc theorem to A^T. Then all eigenvalues of A are contained in the union of another n discs

$$\bigcup_{j=1}^{n} \{z \in \mathbf{C} : |z - a_{jj}| \leq C_j(A)\} = G(A^T).$$

Let $p_1, p_2, p_3, \ldots, p_n$ be positive real numbers and

$$D = \text{diag}(p_1, p_2, p_3, \ldots, p_n).$$

The Geršgorin disc theorem applies to the matrix $D^{-1}AD$, the eigenvalues of A lie in the region

$$\bigcup_{i=1}^{n}\{z \in \mathbf{C} : |z - a_{ii}| \leq \frac{1}{p_i} \sum_{i=1, i\neq j}^{n} p_j |a_{ij}|\} = G(D^{-1}AD),$$

and

$$\bigcup_{j=1}^{n}\{z \in \mathbf{C} : |z - a_{jj}| \leq p_i \sum_{i=1, i\neq j}^{n} \frac{1}{p_j} |a_{ij}|\} = G((D^{-1}AD)^T).$$

Suppose that $F(x, y, s) = 0$ represents an one-parameter family of curves in the xy-plane with parameter s. Frequently there is a curve Γ in the plane with the properties: (i) each curve of the family is tangent to Γ; (ii) at each point p of Γ there is a curve in family which is tangent to Γ at p. If such a curve Γ exists it is called the envelope of the family. Usually x and y can be solved by the two equations $F(x, y, s) = 0$ and $\frac{\partial F}{\partial s}(x, y, s) = 0$ to yeild the parametric equation of the envelope. The idea using envelopes that treats the numerical range of a matrix can be found in [2].

Theorem 1. *Let*

$$A = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \in M_3(\mathbf{C})$$

be an upper triangular nilpotent matrix. Assume α is the maximum of $\{|a|, |b|, |c|\}$, and β is the second largest of $\{|a|, |b|, |c|\}$. Then

$$\text{area}(W(A)) \leq \frac{(240\alpha^2 + 624|\alpha\beta| + 275\beta^2)\pi}{512}. \tag{2}$$

Proof. For each $0 \leq \theta < 2\pi$, define

$$H_\theta = \frac{1}{2}(e^{i\theta}A + e^{-i\theta}A^*)$$

and

$$D = \text{diag}(1 + \cos^2\theta, (1 + \cos^2\theta)^2, (1 + \cos^2\theta)^3).$$

Then

$$M_\theta \equiv D^{-1}H_\theta D = \begin{pmatrix} 0 & \frac{1}{2}ae^{i\theta}(1 + \cos^2\theta) & \frac{1}{2}be^{i\theta}(1 + \cos^2\theta)^2 \\ \frac{1}{2}\frac{\overline{a}e^{i\theta}}{1+\cos^2\theta} & 0 & \frac{1}{2}ce^{i\theta}(1 + \cos^2\theta) \\ \frac{1}{2}\frac{\overline{b}e^{-i\theta}}{(1+\cos^2\theta)^2} & \frac{1}{2}\frac{\overline{c}e^{-i\theta}}{1+\cos^2\theta} & 0 \end{pmatrix},$$

and

$$\begin{aligned}
R_1(M_\theta) &= \frac{1}{2}(|a|(1+\cos^2\theta) + |b|(1+\cos^2\theta)^2)\\
R_2(M_\theta) &= \frac{1}{2}(\frac{|a|}{1+\cos^2\theta} + |c|(1+\cos^2\theta))\\
R_3(M_\theta) &= \frac{1}{2}(\frac{|b|}{(1+\cos^2\theta)^2} + \frac{|c|}{1+\cos^2\theta})\\
C_1(M_\theta) &= \frac{1}{2}(\frac{|b|}{(1+\cos^2\theta)^2} + \frac{|a|}{1+\cos^2\theta})\\
C_2(M_\theta) &= \frac{1}{2}(\frac{|c|}{1+\cos^2\theta} + |a|(1+\cos^2\theta))\\
C_3(M_\theta) &= \frac{1}{2}(|c|(1+\cos^2\theta) + |b|(1+\cos^2\theta)^2).
\end{aligned}$$

Suppose that $\sigma(M_\theta) = \{\lambda_1(M_\theta) \geq \lambda_2(M_\theta) \geq \lambda_3(M_\theta)\}$. Since

$$\sigma(M_\theta) \subseteq G(M_\theta) \bigcap G(M_\theta^T),$$

it follows that

$$\lambda_1(M_\theta) \leq \min\{\max_{i=1,2,3} R_i(M_\theta), \max_{i=1,2,3} C_i(M_\theta)\}.$$

We consider the following three conditions on a,b,c.

Case i) If $|c| \leq \min(|a|, |b|)$, then

$$\max\{R_1(M_\theta), R_2(M_\theta), R_3(M_\theta)\} = R_1(M_\theta)$$

and by [8]

$$l_\theta = \{z \in \mathbf{C},\ \text{Re}\ z = R_1(M_\theta)\}$$

is the right vertical supporting line of the set $W(e^{i\theta}A)$. The rectangular coordinate system for the line $e^{-i\theta}l_\theta$ satisfies

$$\frac{y + R_1(M_\theta)\sin\theta}{x - R_1(M_\theta)\cos\theta} = \cot\theta.$$

Then we get

$$x\cos\theta - y\sin\theta = R_1(M_\theta). \tag{3}$$

Differentiate (3) with respect to θ,

$$x\sin\theta + y\cos\theta = -R_1'(M_\theta). \tag{4}$$

From equations (3) and (4), we solve x and y in terms of θ to yield a parametric equation of the envelope

$$x(\theta) = R_1(M_\theta)\cos\theta - R_1'(M_\theta)\sin\theta \tag{5}$$

$$y(\theta) = -R_1(M_\theta)\sin\theta - R_1'(M_\theta)\cos\theta. \tag{6}$$

From (5), (6) we have

$$x(\theta) = \frac{\cos\theta}{16}(20|a| + 39|b| - 4|a|\cos 2\theta - 4|b|\cos 2\theta - 3|b|\cos 4\theta) \tag{7}$$

$$y(\theta) = \frac{\sin\theta}{16}(-4|a| + 9|b| + 4|a|\cos 2\theta + 20|b|\cos 2\theta + 3|b|\cos 4\theta). \tag{8}$$

It is easy to vertify that (7) and (8) are symmetric with respect to x-coordinate. Then we have

$$\begin{aligned}
\operatorname{area}(W(A)) &\le 2\int y\,dx \\
&= 2\int_0^\pi \frac{\sin^2\theta}{512}(114a^2 + 320|ab| + 423b^2 \\
&\quad -192a^2\cos 2\theta - 608|ab|\cos 2\theta + 296b^2\cos 2\theta \\
&\quad +48a^2\cos 4\theta + 192|ab|\cos 4\theta + 876b^2\cos 4\theta \\
&\quad +96|ab|\cos 6\theta + 408b^2\cos 6\theta + 45b^2\cos 8\theta)d\theta \\
&= \frac{(240a^2 + 624|ab| + 275b^2)\pi}{512}.
\end{aligned}$$

On the other hand, we consider

$$D = \operatorname{diag}((1+\cos^2\theta)^3, 1+\cos^2\theta, (1+\cos^2\theta)^2).$$

Then

$$\max\{C_1(M_\theta), C_2(M_\theta), C_3(M_\theta)\} = C_1(M_\theta) = \frac{1}{2}(|b|(1+\cos^2\theta) + |a|(1+\cos^2\theta)^2).$$

Similarly, we obtain

$$\operatorname{area}(W(A)) \le \frac{(240b^2 + 624|ab| + 275a^2)\pi}{512}.$$

Case ii) $|a| \le \min(|c|, |b|)$ and Case iii) $|b| \le \min(|a|, |c|)$ can be treated in a similar way.

3. Newton's Method

We briefly describe the Newton's method, for details, see for example [3]. Let $f(x)$ be a function and x_0 be a given initial point.

$$\begin{aligned} x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\ &\vdots \\ x_n &= x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}. \end{aligned} \tag{9}$$

The sequence x_n is approximating to a zero $\hat{x}$ of $f(x) = 0$ when x_0 is sufficiently close to $\hat{x}$.

Let $A \in M_3(\mathbf{R})$ be an upper triangular nilpotent matrix

$$A = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$H_\theta = \frac{1}{2}(Ae^{i\theta} + A^* e^{-i\theta}) = \begin{pmatrix} 0 & \frac{1}{2}ae^{i\theta} & \frac{1}{2}be^{i\theta} \\ \frac{1}{2}ae^{-i\theta} & 0 & \frac{1}{2}ce^{i\theta} \\ \frac{1}{2}be^{-i\theta} & \frac{1}{2}ce^{-i\theta} & 0 \end{pmatrix},$$

and the characteristic polynomial of H_θ is

$$t^3 - pt - q\cos\theta,$$

where $p = \frac{1}{4}(a^2 + b^2 + c^2)$, $q = \frac{1}{4}abc$. Next we apply Newton's method to find an upper bound for the maximum eigenvalue of the matrix H_θ.

Let $f(t) = t^3 - pt - q\cos\theta$, then $f'(t) = 3t^2 - p$ and $f''(t) = 6t$. We choose an initial point

$$x_0 = \sqrt{\frac{p+1}{3}} + |q|$$

for Newton's method. It is easy to see that $f(t)$ has maximum root in the interval $[\sqrt{\frac{p}{3}}, \infty)$ since $f'(\frac{p}{3}) = 0$ and $f(t)$ is concave upward in the interval $[0, \infty)$. Observe that $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ in (9) is the intersection of the x-axis and the tangent line to the function $f(t)$ at the point x_0. Therefore, x_1 is greater than the maximum root of $f(t)$. We compute that

$$x_1 = x_0 - \frac{x_0^3 - px_0}{3x_0^2 - p} - \frac{q\cos\theta}{3x_0^2 - p}.$$

Now, x_1 is greater than the maximum eigenvalue of the matrix H_θ. For $\theta \in [0, 2\pi)$, define the function

$$\phi(\theta) = x_0 - T_1 - T_2 \cos\theta,$$

where $T_1 = \frac{x_0^3 - x_0 p}{3x_0^2 - p}$ and $T_2 = \frac{q}{3x_0^2 - p}$. Then the boundary generating curve (5)-(6) of $W(A)$ is contained in the following parametric curve:

$$x(\theta) = (x_0 - T_1)\cos\theta - T_2$$
$$y(\theta) = -x_0 \sin\theta + T_1 \sin\theta.$$

It is clear that $(x(\theta), y(\theta))$ is symmetric with respect to $y = 0$. Thus

$$\begin{aligned} \text{area}(W(A)) &\le 2\int_0^\pi y(\theta)dx(\theta) \\ &= 2\int_0^\pi (-T_1 + x_0)^2 \sin^2\theta d\theta \\ &= (x_0 - T_1)^2\pi. \end{aligned} \tag{10}$$

The following result is immediately from (10).

Theorem 2. *Let*

$$A = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \in M_3(\mathbf{R})$$

be an upper triangular nilpotent matrix. Then

$$\text{area}(W(A)) \le (x_0 - \frac{x_0^3 - \frac{1}{4}x_0(a^2 + b^2 + c^2)}{3x_0^2 - \frac{1}{4}(a^2 + b^2 + c^2)})^2\pi,$$

where $x_0 = \sqrt{\frac{\frac{1}{4}(a^2+b^2+c^2)+1}{3}} + |\frac{1}{4}abc|$.

4. Johnson's Method

Let $W'(A)$ denote the convex hull of the numerical ranges of the n different $(n-1)$-by-$(n-1)$ principal submatrices of A. Johnson [6] proved that the relative area

$$\frac{\text{area}(W'(A))}{\text{area}(W(A))} \ge \frac{c^2}{6 - 6c + c^2}, \tag{11}$$

where the number c depends only on n and not on the matrix A and the vaule $c = \sqrt{\frac{n-2}{n}}$.

Theorem 3. *Let*

$$A = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \in M_3(\mathbf{C})$$

be an upper triangular nilpotent matrix with a, b, c, not all zero. Then

$$\text{area}(W(A)) \leq \frac{(19 - 6\sqrt{3})}{4} \pi r^2,$$

where $r = \max\{|a|, |b|, |c|\}$.

Proof. The three 2-by-2 principal submatrices of A are

$$A_1 = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}.$$

By [9], $W(A_1)$, $W(A_2)$, $W(A_3)$ are circular discs centered at 0 with respective radius $\frac{|c|}{2}$, $\frac{|b|}{2}$, $\frac{|a|}{2}$. Suppose $r = \max\{|a|, |b|, |c|\}$, then

$$\text{area}(W'(A)) = \text{area}(\text{conv}(\cup_{i=1}^3 W(A_i))) = \frac{r^2}{4}\pi. \tag{12}$$

Substituting (12) and $c = \sqrt{\frac{1}{3}}$ into (11), we have

$$\frac{\frac{r^2}{4}\pi}{\text{area}(W(A))} \geq \frac{1}{19 - 6\sqrt{3}},$$

thus

$$\text{area}(W(A)) \leq \frac{(19 - 6\sqrt{3})}{4} \pi r^2.$$

Remark 1. The idea of the proof of Theorem 3 can be extended to find an upper bound for the area of the numerical range of general upper triangular nilpotent matrix

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & 0 & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & 0 \end{pmatrix} \in M_n(\mathbf{C}).$$

It is easy to deduce that

$$\text{area}(W(A)) \leq (\frac{6-6c_n+c_n^2}{c_n^2})\cdots(\frac{6-6c_3+c_3^2}{c_3^2})(\frac{r}{2})^2\pi,$$

where $c_i=\sqrt{\frac{i-2}{i}}$, $i=3\ldots n$ and $r=\max_{1\leq i,j\leq n}|a_{ij}|$.

Remark 2. Let $A\in M_n(\mathbf{C})$, and let $H_\theta(A)=(1/2)(e^{i\theta}A+e^{-i\theta}A^*)$ be the Hermitian part of the matrix $e^{i\theta}A$, $0\leq\theta<2\pi$. The point

$$p_\theta = x_\theta^* A x_\theta$$

lies on the boundary of $W(A)$, where x_θ is a unit eigenvector of $H_\theta(A)$ corresponding to the maximum eigenvalue $\lambda_{\max}(H_\theta(A))$. Choose $\theta_1,\theta_2,\ldots,\theta_m$, then compute the m boundary points $p_{\theta_1},\ldots,p_{\theta_m}$ of $W(A)$ in counterclockwise order. The m-side polygon formed by $p_{\theta_1},\ldots,p_{\theta_m}$ is inscribed in $W(A)$. The numerical area of $W(A)$ is approximated by the area of the m-side polygon which can be evaluated by

$$\frac{1}{2}\text{Im}(\bar{p}_{\theta_1}p_{\theta_2}+\bar{p}_{\theta_2}p_{\theta_3}+\ldots+\bar{p}_{\theta_{m-1}}p_{\theta_m}+\bar{p}_{\theta_m}p_{\theta_1}).$$

This numerical approach can be found in [1], the numerical value of the area will be compared with other methods in the following example.

We give an example to compare the upper bounds of the area of the numerical range among the methods described above.

Example. Consider

$$A=\begin{pmatrix}0&1&2\\0&0&1\\0&0&0\end{pmatrix}\in M_3(\mathbf{R}).$$

We plot the boundary points according to Chien [1] in figure 1.

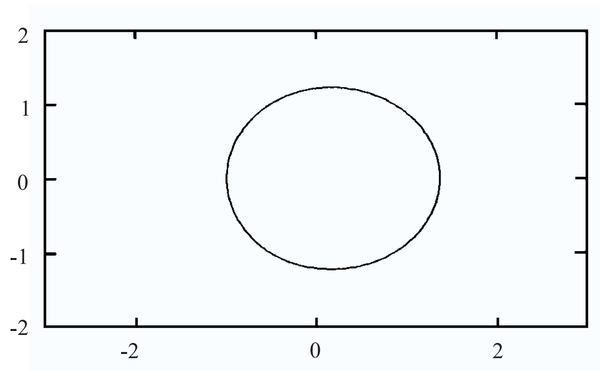

Figure 1.

Chien's method: Following the algorithm described in Remark 2, we calculate that

$$\text{area}(W(A)) \approx 4.5475.$$

Geršgorin disc method: Clearly $\alpha = 2$, $\beta = 1$ in Theorem 1, the inequality (2) gives

$$\text{area}(W(A)) \leq \frac{2483}{512}\pi \approx 15.2354.$$

Newton's method: The initial point

$$x_0 = \frac{1}{2} + \sqrt{\frac{5}{6}},\ T_1 = \frac{45 + \sqrt{30}}{18(7 + \sqrt{120})}.$$

Substituting these into (10), we have

$$\text{area}(W(A)) \leq \frac{(99 + 19\sqrt{30})^2}{81(7 + \sqrt{120})^2}\pi \approx 4.96136.$$

Johnson's method: In Theorem 3, $r = 2$, and we have

$$\text{area}(W(A)) \leq (19 - 6\sqrt{3})\pi \approx 27.041.$$

From this example, we see that the Newton's method is the best estimation for area(W(A)) comparing with Chien's numerical value. The Johnson's method is rough because of the small dimension, $n = 3$, in (11). The Geršgorin disc method is rough too. This is due to the off diagonals of A are relatively large which will be used to count the radii of Geršgorin discs. However, we don't know which one is sharp among the three methods for general case.

5. Idempotent and Special Types

A matrix $A \in M_n(\mathbf{C})$ is idempotent if $A^2 = A$. It is known (cf. [2]) that every nonzero idempotent matrix is unitarily equivalent to

$$\begin{pmatrix} I_m & B \\ 0 & 0 \end{pmatrix} \oplus I_{l-m} \oplus 0, \tag{13}$$

where $m = \text{rank}(AA^* - I)$, $l = \text{rank}(A)$ and $B = \text{diag}(b_1,\ b_2,\ \cdots,\ b_m)$ with $b_1 \geq b_2 \geq \cdots \geq b_m$.

Chien [2] introduced the envelope of the family of the vertical supporting lines, and determined that the numerical range of an idempotent matrix is an elliptic disc. It is also shown that the ellipse satisfies

$$\frac{(x - \frac{1}{2})^2}{\frac{1}{4}(1 + b_1^2)} + \frac{y^2}{\frac{1}{4}b_1^2} = 1.$$

Consequently, we have the following result:

Theorem 4. *Let A be the idempotent matrix defined by* (13)*. Then $W(A)$ is an elliptic disc with foci $(1,0)$ and $(0,0)$, semi-major axis of length $\frac{1}{2}\sqrt{1+b_1^2}$ and semi-minor axis of legth $\frac{1}{2}b_1$. The area of $W(A)$ is $\frac{\pi}{4}b_1\sqrt{1+b_1^2}$.*

Consider the special matrix

$$A_n = \begin{pmatrix} 0 & a_1 & 0 & 0 & \ldots & 0 & 0 \\ b_1 & 0 & a_2 & a_3 & \ldots & a_{n-2} & a_{n-1} \\ 0 & b_2 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & b_{n-2} & 0 & 0 & \ldots & 0 & 0 \\ 0 & b_{n-1} & 0 & 0 & \ldots & 0 & 0 \end{pmatrix} \in M_n(\mathbf{R}). \tag{14}$$

We denote

$$H_\theta(A_n) = \frac{1}{2}(e^{i\theta}A_n + e^{-i\theta}A_n^*);$$

$$\begin{aligned} P_{H_\theta(A_n)}(t) &= \text{the characteristic polynomial of } H_\theta(A_n); \\ \gamma_i(\theta) &= (a_i e^{i\theta} + b_i e^{-i\theta})(b_i e^{i\theta} + a_i e^{-i\theta}); \\ \alpha_n &= \sum_{i=1}^{n-1} a_i b_i; \\ \beta_n &= \sum_{i=1}^{n-1} a_i^2 + b_i^2. \end{aligned}$$

Lemma. *Let A_n be the matrix defined by* (14)*. Then*

$$P_{H_\theta(A_n)}(t) = (-t)^n - \frac{1}{4}(-t)^{n-2} \sum_{i=1}^{n-1} \gamma_i(\theta), n \geq 2,$$

where

$$\gamma_i(\theta) = (a_i e^{i\theta} + b_i e^{-i\theta})(b_i e^{i\theta} + a_i e^{-i\theta}).$$

Proof. We prove by induction. For $n = 2$,

$$\begin{aligned} P_{H_\theta(A_2)}(t) &= \det \begin{pmatrix} -t & \frac{1}{2}(a_1 e^{i\theta} + b_1 e^{-i\theta}) \\ \frac{1}{2}(b_1 e^{i\theta} + a_1 e^{-i\theta}) & -t \end{pmatrix} \\ &= (-t)^2 - \frac{1}{4}(a_1 e^{i\theta} + b_1 e^{-i\theta})(b_1 e^{i\theta}) + a_1 e^{-i\theta} \\ &= (-t)^2 - \frac{1}{4}\gamma_1(\theta). \end{aligned}$$

Now we assume that (15) holds for a positive integer p, i.e.

$$P_{H_\theta(A_p)}(t) = (-t)^p - \frac{1}{4}(-t)^{p-2} \sum_{i=1}^{p-1} \gamma_i(\theta).$$

Consider the $p + 1$ dimension,

$$A_{p+1} = \begin{pmatrix} 0 & a_1 & 0 & 0 & \dots & 0 & 0 \\ b_1 & 0 & a_2 & a_3 & \dots & a_{p-1} & a_p \\ 0 & b_2 & 0 & 0 & \dots & 0 & 0 \\ 0 & b_3 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \ddots & \vdots & \vdots \\ 0 & b_p & 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Then

$$
\begin{aligned}
&P_{H_\theta(A_{p+1})}(t) = \det[H_\theta(A_{p+1}) - tI] \\
&= \det\begin{pmatrix}
-t & \frac{(a_1e^{i\theta}+b_1e^{-i\theta})}{2} & 0 & \dots & 0 \\
\frac{(b_1e^{i\theta}+a_1e^{-i\theta})}{2} & -t & \frac{(a_2e^{i\theta}+b_2e^{-i\theta})}{2} & \dots & \frac{(a_pe^{i\theta}+b_pe^{-i\theta})}{2} \\
0 & \frac{(b_2e^{i\theta}+a_2e^{-i\theta})}{2} & -t & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \frac{(b_{p-1}e^{i\theta}+a_{p-1}e^{-i\theta})}{2} & 0 & \dots & 0 \\
0 & \frac{(b_pe^{i\theta}+a_pe^{-i\theta})}{2} & 0 & \dots & -t
\end{pmatrix} \\
&= (-t)\det(H_\theta(A_p)) + \frac{1}{2}(-1)^{p+3}(b_pe^{i\theta} + a_pe^{-i\theta}) \\
&\quad \times\det\begin{pmatrix}
-t & 0 & \dots & 0 & 0 \\
\frac{(b_1e^{i\theta}+a_1e^{-i\theta})}{2} & \frac{(a_2e^{i\theta}+b_2e^{-i\theta})}{2} & \dots & \frac{(a_{p-1}e^{i\theta}+b_{p-1}e^{-i\theta})}{2} & \frac{(a_pe^{i\theta}+b_pe^{-i\theta})}{2} \\
0 & -t & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \\
0 & 0 & \dots & -t & 0
\end{pmatrix} \\
&= (-t)\det(H_\theta(A_p)) - \frac{1}{4}(-1)^{2p+4}(-t)^{p-1}\gamma_p(\theta) \\
&= (-t)[(-t)^p - \frac{1}{4}(-t)^{p-2}\sum_{i=1}^{p-1}\gamma_i(\theta)] - \frac{1}{4}(-t)^{p-1}\gamma_p(\theta) \\
&= (-t)^{p+1} - \frac{1}{4}(-t)^{p-1}\sum_{i=1}^{p}\gamma_i(\theta).
\end{aligned}
$$

Theorem 5. Let A_n be the matrix defined by (14). If there exist i and j such that $a_i + b_i \neq 0$ and $a_j - b_j \neq 0$, then $W(A_n)$ is an elliptic disc with ellipse

$$\frac{x^2}{(\frac{\sqrt{\sum_{i=1}^{n-1}(a_i+b_i)^2}}{2})^2} + \frac{y^2}{(\frac{\sqrt{\sum_{j=1}^{n-1}(a_j-b_j)^2}}{2})^2} = 1,$$

and

$$\text{area}(W(A_n)) = \frac{\pi}{4}\Big(\sum_{i=1}^{n-1}(a_i+b_i)^2 \cdot \sum_{j=1}^{n-1}(a_j+b_j)^2\Big)^{\frac{1}{2}}.$$

Proof. By Lemma,

$$P_t(H_\theta(A_n)) = (-t)^n - \frac{1}{4}(-t)^{n-2}\sum_{i=1}^{n-1}\gamma_i(\theta).$$

The eigenvalue of $H_\theta(A_n)$ are 0, $\pm\frac{\sqrt{2\alpha_n\cos 2\theta+\beta_n}}{2}$, and its maximum eigenvalue is $\frac{1}{2}\sqrt{2\alpha_n\cos 2\theta+\beta_n}$, where

$$\begin{aligned}
\gamma_i(\theta) &= (a_i e^{i\theta} + b_i e^{-i\theta})(b_i e^{i\theta} + a_i e^{-i\theta});\\
\alpha_n &= \sum_{i=1}^{n-1} a_i b_i;\\
\beta_n &= \sum_{i=1}^{n-1} a_i^2 + b_i^2.
\end{aligned}$$

Applying the enevloping method [2], we obtain the parametric equation of the envelope of $W(A_n)$

$$\begin{aligned}
x &= \frac{\sqrt{2\alpha_n\cos 2\theta+\beta_n}\cos\theta}{2} + \frac{\alpha_n\sin 2\theta\sin\theta}{\sqrt{2\alpha_n\cos 2\theta+\beta_n}}\\
&= \frac{(\beta_n-2\alpha_n)\cos\theta}{2\sqrt{\beta_n+2\alpha_n\cos 2\theta}}\\
y &= -\frac{\sqrt{2\alpha_n\cos 2\theta+\beta_n}\sin\theta}{2} + \frac{\alpha_n\sin 2\theta\cos\theta}{\sqrt{2\alpha_n\cos 2\theta+\beta_n}}\\
&= \frac{-(\beta_n-2\alpha_n)\sin\theta}{2\sqrt{\beta_n+2\alpha_n\cos 2\theta}}.
\end{aligned}$$

Then

$$\frac{x}{\frac{\sqrt{\beta_n+2\alpha_n}}{2}} = \frac{\sqrt{\beta_n+2\alpha_n}\cos\theta}{\sqrt{\beta_n+2\alpha_n\cos 2\theta}}, \tag{15}$$

$$\frac{y}{\frac{\sqrt{\beta_n-2\alpha_n}}{2}} = \frac{-\sqrt{\beta_n-2\alpha_n}\sin\theta}{\sqrt{\beta_n+2\alpha_n\cos 2\theta}}. \tag{16}$$

Square both sides of (16) and (17) and sum them up, we have

$$\frac{x^2}{(\frac{\sqrt{\beta_n+2\alpha_n}}{2})^2} + \frac{y^2}{(\frac{\sqrt{\beta_n-2\alpha_n}}{2})^2} = 1. \tag{17}$$

The area of $W(A_n)$ is obtained from the ellipse equation.

Remark 3. The matrix considered in (14) has non-zero entries in row 2 and column 2. We may permute the matrix (14) so that it has non-zero entries in row k and column k. Thus the result of Theorem 5 is valid for matrices of the form with non-zero entries in row k and column k.

References

[1] M. T. Chien, *Numerical computation and circularity of higher numerical range,* Soochow Journal of Mathematics, 20(1994), 555-567.

[2] M. T. Chien, *The envelope of the generalized numerical range*, Linear and Multilinear Algebra, 43(1998), 363-376.

[3] L. J. Goldstein, D. C. Lay and D. I. Schneider, Calculus and Its Applications, New Jersey, 1996.

[4] R. A. Horn and C. R. Johnson, Matrix Analysis, Cambridge University Press, 1985.

[5] R. A. Horn and C. R. Johnson, Topics in Matrix Analysis, Cambridge University Press, New York, 1991.

[6] C. R. Johnson, *Numerical ranges of principal submatrices,* Liear Algebra and Its Applications, 37(1981), 23-24

[7] D. S. Keeler, L. Rodman and I. M. Spitkovsky, *The numerical range of* 3×3 *matrices,* Linear Algebra and Its Applications, 252(1997), 112-139.

[8] C. K. Li, C. H. Sung and N. K. Tsing. *c-convex matrix, characterizations, inclusion relations and normality,* Linear and Multilinear Algebra, 25(1989), 275-287.

[9] M. Marcus and C. Pesce, *Computer generated numerical ranges and some resulting theorems,* Linear and Multilinear Algebra, 20(1987), 121-157.

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