ON CLOSE-TO-CONVEX FUNCTIONS OF COMPLEX ORDER

BY

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Abstract. Using the concept of Ruscheweyh derivatives, the class $K_n(b)$, for $b \neq 0$ and complex, $n$ nonnegative integer, was defined by Al-Amiri and Fernando. In this paper an inclusion result, a covering theorem and the analogue of Polya-Schoenberg conjecture for $K_n(b)$ are proved. Some other interesting properties of this class are also discussed.

1. Introduction

Let $A$ denote the class of functions $f : f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, analytic in the unit disk $E = \{ z : |z| < 1 \}$. The Hadamard product (or convolution) of two functions $f, g \in A$ is denoted by $f * g$. Let

$$D^n f = f * \frac{z}{(1 - z)^{n+1}}, n \in N_0 = \{0, 1, 2, 3 \ldots \}.$$ 

The operator $D^n$ is called the Ruscheweyh derivative of order $n$.

In [13], the classes $R_n, n \in N_0$ are defined as follows. A function $f \in R_n$ if and only if $f \in A$ and

$$Re \frac{z(D^n f(z))'}{D^n f(z)} > 0, z \in E.$$ 

It is known ([13]) that

$$R_{n+1} \subset R_n, n \in N_0 \quad (1.1)$$

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and hence \( f \in R_n \) implies that \( f \in R_0 \equiv S^* \), the class of starlike univalent functions.

We now have the following.

**Definition 1.1.** Let \( f \in A \) and \( b \neq 0 \) (complex) and let

\[
Re\left\{1 + \frac{1}{b} \frac{z(D^n f(z))}{D^n g(z)} - 1\right\} > 0, z \in E,
\]

for some \( g \in R_n, n \in N_0 \).

Then \( f \) is said to belong to the class \( K_n(b) \). The class \( K_n(b) \) is introduced in [2]. Note that, for \( f(z) = g(z), n = 0 \), we have the class of starlike functions of complex order \( b \) defined in [3], and \( K_0(1) \) is the class \( K \) of close-to-convex univalent functions.

2. Preliminary Results

The following lemma is due to Miller [9].

**Lemma 2.1.** Let \( u \) and \( v \) denote complex variables, \( u = u_1 + iv_2 \) and \( v = v_1 + iv_2 \) and let \( \psi(u,v) \) be a complex-valued function that satisfies the conditions

(i) \( \psi(u,v) \) is continuous in a domain \( D \subset C^2 \).

(ii) \((1,0) \in D \) and \( \psi(1,0) > 0 \).

(iii) \( Re(2i_2,v_1) \leq 0 \) whenever \((i_2,v_1) \in D \) and \( v_1 \leq -\frac{1}{2}(1 + u_2^2) \).

If \( h(z) = 1 + \sum_{k=2}^{\infty} c_k z^k \) is a function that is analytic in \( E \), such that \((h(z),zh'(z)) \in D \) and \( Re\psi(h(z),zh'(z)) > 0 \) holds for all \( z \in E \), then \( Re h(z) > 0 \) in \( E \).

**Lemma 2.2.** Let \( f : f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in K_n(b), n \in N_0 \). Then

\[
|a_k| \leq \frac{n!(k-1)!}{(n+k-1)!} ((k-1)|b| + 1).
\]

This result is sharp and the extremal function \( f_0 \) is given by

\[
f_0(z) = z + \sum_{k=2}^{\infty} \frac{n!(k-1)!}{(n+k-1)!} ((k-1)b + 1) z^k.
\]
Lemma 2.3. Let $n \in N_0$. If $f \in K_n(b)$, then $f \in K_n(1)$ for $|z| < r_1$ where
\begin{equation}
    r_1 = \frac{1}{|b| + \sqrt{|b|^2 - 2Re b} + 1}.
\end{equation}

The sharpness of the result is given by the function $f_1 \in K_n(b)$ defined by
\begin{equation}
    (D^n f_1(z))' = \frac{1 + (2b - 1)z}{(1 - z)^3}.
\end{equation}

For the Lemmas 2.2 and 2.3 we refer to [2].

Lemma 2.4. ([12]) Let $\phi$ be convex and $g$ starlike in $E$. Then, for $F$ analytic in $E$ with $F(0) = 1$, $\frac{\phi \circ F - \phi \circ g}{\phi \circ g}(E)$ is contained in the convex hull of $F(E)$.

3. Main Results

In the following, we prove that all functions in $K_n(b)$ are close-to-convex of complex order $b$.

Theorem 3.1. $K_{n+1}(b) \subset K_n(b)$ for each $n \in N_0$.

Proof. Let $f \in K_{n+1}(b)$. Then
\[
    Re\{1 + \frac{1}{b}[z(D^{n+1}f(z))' - 1]\} > 0, \quad z \in E
\]
for some $g \in R_{n+1}$.

We define an analytic function $h(z)$ in $E$ such that
\begin{equation}
    1 + \frac{1}{b}[z(D^n f(z))' - 1] = h(z),
\end{equation}
where $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$.

We shall show that $Re \ h(z) > 0$ for $z \in E$.

From (3.1) we can write
\begin{equation}
    z(D^n f(z))' = D^n g(z)[bh(z) + (1 - b)].
\end{equation}

Now, using a well-known identity
\begin{equation}
    z(D^n f(z))' = (n + 1)D^{n+1} f(z) - nD^n f(z)
\end{equation}
in (3.2), we have
\[
\frac{z(D^{n+1}f(z))'}{D^{n+1}g(z)} = \frac{D^{n+1}(z f'(z))}{D^{n+1}g(z)}
= \frac{\frac{z[D^n(z f'(z))']}{D^n g(z)} + n \frac{D^n [z f'(z)]}{D^n g(z)}}{\frac{z[D^n g(z)]'}{D^n g(z)} + n}.
\] (3.4)

Since \( g \in R_{n+1} \subset R_n \), we can write
\[
\frac{z(D^n g(z))'}{D^n g(z)} = H(z), \quad Re H(z) > 0, \quad z \in E.
\] (3.5)

Also, differentiating (3.2), we have
\[
\frac{z(z(D^n f(z))')}{D^n g(z)} = bh'(z) + [bh(z) + (1 - b)] H(z).
\] (3.6)

Using (3.5) and (3.6) in (3.4), we obtain
\[
1 + \frac{1}{b} \left[ \frac{z(D^{n+1}f(z))'}{D^{n+1}g(z)} - 1 \right] = h(z) + \frac{zh'(z)}{H(z) + n}.
\] (3.7)

We form the function \( \psi(u, v) \) by taking \( u = h(z), \quad v = zh'(z) \) in (3.7) as
\[
\psi(u, v) = u + \frac{v}{H(z) + n}.
\] (3.8)

It is clear that the function \( \psi(u, v) \) defined by (3.8) satisfies conditions (i) and (ii) of Lemma 2.1. To verify the condition (iii), we proceed as follows.
\[
Re \, \psi(iu_2, v_1) = \frac{v_1(h_1 + n)}{(h_1 + n)^2 + h_2^2},
\]
where \( H(z) = h_1 + ih_2, \quad h_1 \) and \( h_2 \) being the functions of \( x \) and \( y \) and \( Re H(z) = h_1 > 0, \quad z \in E. \)

By putting \( v_1 \leq -\frac{1}{2}(1 + u_2^2) \), we have
\[
Re \, \psi(iu_2, v_1) \leq -\frac{(1 + u_2^2)(h_1 + n)}{2[(h_1 + n)^2 + h_2^2]} \leq 0.
\]

Hence, applying Lemma 2.1, we have \( Re h(z) > 0 \) for \( z \in E \) and consequently \( f \in K_n(b). \)
Remark 3.1.
(i) From Theorem 3.1 and Lemma 2.3 we deduce that $K_n(b)$ consists of close-to-convex (and hence univalent) functions for $b > 1/2$.
(ii) The radius of univalence for $f \in K_n(b)$ is $r_1$ given by (2.2).

We now prove a covering result.

Theorem 3.2. Let $f \in K_n(b)$ for $b > 1/2$. If $B$ is the boundary of the image of $E$ under $f$, then every point of $B$ has a distance of at least $\frac{n+1}{2n+|b|+3}$ from the origin. This result is sharp.

Proof. Let $f(z) \neq c, c \neq 0$.

Since $f \in K_n(b)$, for $b > 1/2$ is univalent in $E$, so is $\frac{cf(z)}{c-f(z)}$.

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then

$$\frac{cf(z)}{c-f(z)} = z + (a_2 + \frac{1}{c})z^2 + \ldots,$$

and

$$|a_2 + \frac{1}{c}| \leq 2,$$

which gives us

$$\left|\frac{1}{c}\right| \leq 2 + \frac{|b| + 1}{n + 1} = \frac{2n + |b| + 3}{n + 1},$$

by using Lemma 2.2. The sharpness follows from the function $f_0$ defined by (2.1).

Theorem 3.3 Let $F \in K_n(b)$ and let $f$ be defined as

$$f(z) = (n+1)z^{-n} \int_0^z t^{n-1}F(t)dt. \quad (3.9)$$

Then $f \in K_{n+1}(b)$.

Proof. Let $g(z) = (n+1)z^{-n} f_0^{\frac{z}{n}} t^{n-1}G(t)dt$, and let $G \in R_n$. Then it is known ([1]) that $f \in R_{n+1}$.

Now, from (3.9), we have

$$D^nF(z) = \frac{n}{n+1}D^n f(z) + \frac{1}{n+1}z(D^n f(z))'.$$  \quad (3.10)
Using (3.3) and (3.10), we have
\[ D^n F(z) = D^{n+1} f(z). \]
Hence
\[ 1 + \frac{1}{b} \left[ \frac{z(D^{n+1} f(z))'}{D^{n+1} g(z)} - 1 \right] = 1 + \frac{1}{b} \left[ \frac{z(D^n f(z))'}{D^n g(z)} - 1 \right], \]
and the result follows immediately from the fact that \( F \in K_n(b) \).

We can easily prove the following:

**Theorem 3.4.** For \( n \in N_0 \), \( \bigcap_{n=0}^{\infty} K_n(b) = \{ id \} \), where \( id \) is the identity function \( z \).

In 1973, Ruscheweyh and Schild-Small [12] proved the Polya-Schoenberg conjecture namely if \( \phi \) is convex and \( f \in S^* \) or \( K \), then \( f \ast \phi \) also belong to \( S^* \) or \( K \) respectively. We shall prove the analogue of this conjecture for the class \( K_n(b) \) and give some interesting applications of this result.

**Theorem 3.5.** Let \( f \in K_n(b) \) and \( \phi \) be convex univalent in \( E \). Then \( f \ast \phi \in K_n(b) \).

**Proof.** Since it is known ([1]) that, for \( g \in R_n \), \( g \ast \phi \in R_n \), it is sufficient to prove that
\[ \text{Re} \left[ 1 + \frac{1}{b} \left\{ \frac{z D^n (f \ast \phi)'}{D^n (g \ast \phi)} - 1 \right\} \right] > 0, \quad z \in E, \quad b \neq 0. \]
Now
\[ 1 + \frac{1}{b} \left\{ \frac{z D^n (f \ast \phi)'}{D^n (g \ast \phi)} - 1 \right\} = \frac{\phi \ast \left[ 1 + \frac{1}{b} \left\{ \frac{z D^n (f')}{D^n (g')} - 1 \right\} \right] D^n g}{\phi \ast D^n g}, \]
where \( \text{Re} \ F(z) > 0 \) for \( z \in E \) and \( D^n g \in S^* \).
Hence, using Lemma 2.4, we conclude that \( \text{Re} \frac{\phi \ast F D^n g}{\phi \ast D^n g} > 0 \), \( z \in E \) which implies \( \phi \ast f \in K_n(b) \).

**4. Applications of Theorem 3.6**

Let \( f = T(F) \), where \( T \) is a linear operator. We shall study the mapping properties of \( f \) where \( F \in K_n(b) \) and \( T \) is a differential or an integral operator.
We first consider the case when $T$ is an integral operator. Let, for $0 < \lambda \leq 1$, $T \equiv I_\lambda : A \to A, I_\lambda(F) = F$,

$$f(z) = \lambda^{-1}z^{1-\frac{1}{\lambda}} \int_0^z t^{\frac{1}{\lambda}-2}F(t)dt.$$  

(4.1)

Many authors have studied this operator when $F$ is in one of the subclasses of univalent functions, see [7, 4] and [6]. We prove the following.

**Theorem 4.1.** Let $F \in K_n(b), n \in N_0$. Then $I_\lambda(F) = f$ defined by (4.1) also belongs to $K_n(b)$.

**Proof.** Let $h_\lambda(z) = \sum_{j=1}^{\infty} \frac{\gamma+1}{\gamma+j}z^j$, $1 + \gamma = \frac{1}{\lambda}$, $\lambda > 0$. Then $h_\lambda(z)$ is convex for $z \in E$, see [11]. Setting

$$I_\lambda(F) = f = (h_\lambda * F)(z),$$

and using Theorem 3.6, we obtain the required result.

Next we investigate the mapping properties of the function $f(z)$ defined by

$$D_\lambda(F(z)) = f(z) = (1 - \lambda)F(z) + \lambda F'(z);$$

(4.2)

where $\lambda > 0$ and $z \in E$.

This differential operator, for $\lambda = 1/2$, was first considered by Livingston [8] and then by many others, see [6]. We now prove:

**Theorem 4.2.** Let $F \in K_n(b), n \in N_0$. Then $D_\lambda(F(z)) = f(z)$ defined by (4.2) belongs to $K_n(b)$ for $|z| < r_0$, where $r_0$ is given by

$$r_0 = \frac{1}{2\lambda + \sqrt{4\lambda^2 - 2\lambda + 1}}.$$  

(4.3)

**Proof.** Consider the function

$$H_\lambda(z) = \frac{z[1 - (1 - \lambda)z]}{(1 - z)^2}.$$  

The radius of convexity for $H_\lambda(z)$ is $r_0$, where $r_0$ is given by (4.3). Now the proof follows immediately from the fact that

$$D_\lambda(F(z)) = f(z) = (H_\lambda * F)(z)$$
and Theorem 3.

References


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