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A CLASS OF CONVOLUTION INTEGRAL EQUATIONS CONCERNING A GENERALIZED POLYNOMIAL SET

BY

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Abstract. In the present work, an attempt has been made to solve a certain class of convolution integral equation of Fredholm type with a generalized polynomial set kernel. By making use of the Mellin transform technique, solution of the integral equation has been established. Through exponentation, our result encompass several cases of interest.

1. Introduction

Agrawal and Chaubey [1, 5] have introduced the polynomial set $R_n(x)$ by means of the following Rodrigues formula

$$R_n^{\alpha,\beta}[x;a,b,c,d;p,q,\nu,\xi;w(x)] \equiv R_n(x) \tag{1}$$
$$= \frac{(ax^p+b)^{-\alpha}(cx^q+d)^{-\beta}}{K_n w(x)} T_{k;\lambda}^n\Big[(ax^p+b)^{\alpha+\nu n}(cx^q+d)^{\beta+\xi n} w(x)\Big], \quad n=0,1,2,\ldots$$

where

$$T_{k;\lambda} = x^k(\lambda + xD_x), \qquad D_x = \frac{d}{dx}, \tag{2}$$

$\{K_n\}_{n=0}^{\infty}$ is a sequence of constants, $a,b,c,d,\alpha,\beta,p,q,\nu,\xi$ are constants, and $w(x)$ is any general function of x, differentiable an arbitrary number of times.

The above mentioned polynomial set (1) is general in nature and yields a number of known polynomials as its special cases. In particular, $p=q=1$, $K_n =$

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$n!$, $\lambda = 0$, $k = -1$, reduces the polynomial set $R_n(x)$ to $S_n^{\alpha,\beta}[x; a, b, c, d; \nu, \xi; w(x)]$, defined by Srivastava and Panda [6].

The aim of the present paper is to obtain the inversion of the integral

$$g(x) = \int_0^\infty h(\frac{x}{y}) f(y) (\frac{dy}{y}), \qquad (x > 0), \tag{3}$$

where g is a prescribed function, f is an unknown function to be determined, and the kernel h is given by

$$\begin{aligned} h(x) &= \frac{(ax^p + b)^\alpha (cx^q + d)^\beta}{e^{tx^r}} K_n R_n^{\alpha,\beta}\left[x; a, b, c, d; p, q, \nu, \xi; e^{-tx^r}\right] \\ &= [x^k(\lambda + xD_x)]^n \left\{(ax^p + b)^{\alpha+\nu n} (cx^q + d)^{\beta+\xi n} e^{-tx^r}\right\}. \end{aligned} \tag{4}$$

2. Mellin Transform of $h(x)$

We begin with the following Lemma:

Lemma. *Let* $H(s) = M\{h(x); s\}$, *where* $h(x)$ *is defined by* (4), *then*

$$\begin{aligned} H(s) = &\sum_{e=0}^{n} \sum_{u=0}^{\alpha+\nu n} \sum_{v=0}^{\beta+\xi n} (-1)^{e+u+v} \frac{(-n)_e}{e!} \frac{(-\alpha - \nu n)_u}{u!} \\ &\cdot \frac{(-\beta - \xi n)_v}{v!} \lambda^{n-e} b^{\alpha+\nu n-u} a^u d^{\beta+\xi n-v} c^v k^e (-\frac{s + kn}{k})_e \\ &\cdot \frac{1}{|r|} \frac{\Gamma(\frac{s+kn+pu+qv}{r})}{(t)^{\frac{s+kn+pu+qv}{r}}}, \end{aligned} \tag{5}$$

provided that $0 < Re(s+kn+pu+qv) < r$, *when* $r > 0$; $r < Re(s+kn+pu+qv) < 0$, *when* $r < 0$; $k \neq 0$ *and* $n, (\alpha + \nu n)$, $(\beta + \xi n) \in N_0$.

Proof. Using the binomial expansions for $(ax^p + b)^{\alpha+\nu n}$, $(cx^q + d)^{\beta+\xi n}$ and $[x^k(\lambda + xD_x)]^n$, we have

$$\begin{aligned} h(x) = &\sum_{e=0}^{n} \sum_{u=0}^{\alpha+\nu n} \sum_{v=0}^{\beta+\xi n} (-1)^{e+u+v} \frac{(-n)_e}{e!} \frac{(-\alpha - \nu n)_u}{u!} \\ &\cdot \frac{(-\beta - \xi n)_v}{v!} \lambda^{n-e} b^{\alpha+\nu n-u} a^u d^{\beta+\xi n-v} c^v \\ &\cdot x^{k(n-e)} (x^{k+1} D_x)^e \left\{x^{pu+qv} e^{-tx^r}\right\}. \end{aligned} \tag{6}$$

Taking Mellin transform of both sides of equation (6) and using the known formulaes [7, p.14, eq.(2.2)]; [3, p.307, eq.(7)], we find that

$$H(s) = \sum_{e=0}^{n} \sum_{u=0}^{\alpha+\nu n} \sum_{v=0}^{\beta+\xi n} (-1)^{e+u+v} \frac{(-n)_e}{e!} \frac{(-\alpha-\nu n)_u}{u!} \frac{(-\beta-\xi n)_v}{v!}$$
$$\cdot \lambda^{n-e} b^{\alpha+\nu n-u} a^u d^{\beta+\xi n-v} c^v k^e (-\frac{s+kn}{k})_e$$
$$\cdot M\Big\{x^{pu+qv} e^{-tx^r}, s+kn\Big\}. \tag{7}$$

Again, making use of [3, p.307, eq.(7)] and the following known result ([3, p.313]),

$$M\Big\{(e^{-ax^h}); s\Big\} = \frac{1}{|h|} a^{-s/h} \Gamma(\frac{s}{h}) \tag{8}$$

we arrive at the desired result (5).

3. Solution of the Integral Equation (3)

Theorem. *Let the Mellin transforms $F(s)$, $G(s)$ and $H(s) \neq 0$ of the functions $f(x)$, $g(x)$ and $h(x)$ defined by (4) exist and be analytic in some infinite strip $\eta < Re(s) < \zeta$ of the complex s-plane. Also suppose that for a fixed $\sigma \in (\eta, \zeta), h^*(x)$ is defined by*

$$h^*(x) = M^{-1}\Big\{H^*(s); x\Big\} = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} x^{-s} H^*(s) ds, \tag{9}$$

where

$$H^*(s) = \Bigg[A^L \frac{\Gamma(-\frac{s}{A})}{\Gamma(-L-\frac{s}{A})} \sum_{e=0}^{n} \sum_{u=0}^{\alpha+\nu n} \sum_{v=0}^{\beta+\xi n} (-1)^{u+v} \frac{(-n)_e}{e!}$$
$$\cdot \frac{(-\alpha-\nu n)_u}{u!} \frac{(-\beta-\xi n)_v}{v!} \lambda^{n-e} b^{\alpha+\nu n-u} a^u d^{\beta+\xi n-v} c^v \frac{k^e}{|r|}$$
$$\cdot t^{-(\frac{s+AL+B+kn+pu+qv}{r})} \frac{\Gamma(1+\frac{s+kn+AL+B}{k}) \Gamma(\frac{s+kn+AL+B+pu+qv}{r})}{\Gamma(\frac{s+kn+AL+B}{k}-e+1)}\Bigg]^{-1}, \tag{10}$$

provided that $0 < Re(s+AL+B+kn+pu+qv) < r$, when $r > 0$; $r < Re(s+AL+B+kn+pu+qv) < 0$, when $r < 0$; $r, k, A \neq 0$ and $n, (\alpha+\nu n)$, $(\beta+\xi n) \in N_0$. Then the integral equation (3) has its solution given by

$$f(x) = x^{-AL-B} \int_0^\infty y^{-1} h^*(\frac{x}{y}) (y^{A+1} D_y)^L \Big\{y^B g(y)\Big\} dy, \tag{11}$$

provided that the integral exists.

Proof. By convolution theorem for Mellin transforms [3, p.308, eq.(14)], equation (3) changes to

$$H(s)F(s) = G(s), \tag{12}$$

where $H(s), F(s)$ and $G(s)$ are Mellin transforms of $h(x), f(x)$ and $g(x)$ respectively. Replacing s in (12) by $(s + AL + B)(A \neq 0, L \in N_0)$, we have

$$F(s + AL + B) = H^*(s)\Big\{A^L\Big(-\frac{s + AL}{A}\Big)_L\Big\}G(s + AL + B), \tag{13}$$

where $H^*(s)$ is given by (10).
Now, making use of [7, p.14, eq.(2.2)] and [3, p.307, eq.(7)], we obtain

$$F(s + AL + B) = H^*(s)M\Big[(y^{A+1}D_y)^L\Big\{y^B g(y)\Big\}; s\Big], \tag{14}$$

and using the known results [3, p.307, eq.(7)] and [3, p.308, eq.(14)] in (14), we find that

$$M\Big\{x^{AL+B}f(x); s\Big\} = M\Big[\int_0^\infty y^{-1}h^*(\frac{x}{y})(y^{A+1}D_y)^L\Big\{y^B g(y)\Big\}dy; s\Big]. \tag{15}$$

Inverting both sides of (15) by using the Mellin inversion Theorem [3, p.307, eq.(1)], we arrive at the desired solution (11).

4. Applications

Since the polynomial set $R_n(x)$ incorporates in itself several classical as well as other polynomials, solutions of a large number of convolution integral equations for the above mentioned polynomials may be obtained by assigning different values to the parameters in $R_n^{\alpha,\beta}(x)$.

For example, if we take $p = q = 1$, $K_n = n!$; $\lambda = 0$, $k = -1$, then we get the following corollary

Corollary 1. *The convolution integral equation*

$$\int_0^\infty y^{-1}h_1(\frac{x}{y})f(y)dy = g(x) \qquad (x > 0), \tag{16}$$

where the kernel

$$h_1(x) = n!(ax+b)^{\alpha}(cx+d)^{\beta}e^{-tx^r}s_n^{\alpha,\beta}[x;a,b,c,d;\nu,\xi;e^{-tx^r}]$$
$$= D_x^n[(ax+b)^{\alpha+\nu n}(cx+d)^{\beta+\xi n}e^{-tx^r}], \tag{17}$$

possesses the solution

$$f(x) = x^{-AL-B}\int_0^{\infty} y^{-1}h_1^*(\frac{x}{y})(y^{A+1}D_y)^L\Big\{y^B g(y)\Big\}dy, \tag{18}$$

provided that the integral exists, and $h_1^*(x)$ *is the Mellin inverse transform of*

$$H_1^*(s) = \Bigg[A^L\frac{\Gamma(-\frac{s}{A})}{\Gamma(-L-\frac{s}{A})}\sum_{u=0}^{\alpha+\nu n}\sum_{v=0}^{\beta+\xi n}\frac{(-1)^{u+v+n}}{|r|}\frac{(-\alpha-\nu n)_u}{u!}\frac{(-\beta-\xi n)_v}{v!}\cdot b^{\alpha+\nu n-u}a^u$$
$$d^{\beta+\xi n-v}c^v\frac{\Gamma(s+AL+B)\Gamma[(s+AL+B-n+u+v)/r]}{\Gamma(s+AL+B-n)t^{(s+AL+B-n+u+v)/r}}\Bigg]^{-1}, \tag{19}$$

provided that $Re(s+AL+B-n+u+v)>0$*, when* $r>0$*,* $Re(s+AL+B-n+u+v)<0$*, when* $r<0$*;* $r, A\neq 0$ *and* $n, (\alpha+\nu n)$*,* $(\beta+\xi n)$*,* $L\in N_0$*.*

Also on setting $a=1$*,* $b=0$*,* $\beta=0$*,* $\xi=0$*,* $\alpha=a'-\ell$*,* $\nu=\ell$*,* $k=1-\ell$*,* $p=1$*,* $r=-1$*,* $\lambda=0$*, we get*

$$R_n^{a'-\ell}\Big[x;1,0,c,d;1,0,\ell,0;e^{-t/x}\Big] = t^n M_n^{\ell}[x,a',t], \tag{20}$$

where $M_n^{\ell}[x,a',t]$ *is known as generalized Bessel polynomial* [2]*. Thus, we have the following result contained in the following corollary*

Corollary 2. *Under the hypothesis of Theorem, the integral equation*

$$\int_0^{\infty} y^{-1}h_2(\frac{x}{y})f(y)dy = g(x) \qquad (x>0), \tag{21}$$

where the kernel

$$h_2(x) = x^{a'-\ell}e^{-t/x}t^n M_n^{\ell}[x,a',t]$$
$$= (x^{2-\ell}D_x)^n\Big\{x^{a'-\ell+\ell n}e^{-t/x}\Big\} \tag{22}$$

possesses the solution

$$f(x) = x^{-AL-B}\int_0^{\infty} y^{-1}h_2^*(\frac{x}{y})(y^{A+1}D_y)^L\Big\{y^B g(y)\Big\}dy, \tag{23}$$

provided that the integral exists, and $h_2^*(x)$ *is the Mellin inverse transform of*

$$H_2^*(s) = \frac{(-1)^n t^{-s-n-a'+\ell-AL-B}\Gamma(-L-\frac{s}{A})\Gamma(\frac{s+AL+B+n-2\ell n}{\ell}+1)}{\ell^n A^L \Gamma(-\frac{s}{A})\Gamma(\frac{s+AL+B+n-\ell n}{\ell}+1)\Gamma(-s-AL-B-a'+\ell-n)}, \quad (24)$$

provided that $Re(s+AL+B+a') < \ell - n$, $Re(t) > 0$, n, $L \in N_0$ *and* $A \neq 0$. *Now applying the Mellin inversion formula* (9), *and replacing* s *by* $-s$, *we get*

$$h_2^*(x) = \frac{(-1)^n t^{-(n+a'-\ell+AL+B)}}{\ell^n A^L}\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} x^s t^s \frac{\Gamma(-L+\frac{s}{A})}{\Gamma(\frac{s}{A})}$$
$$\cdot\frac{\Gamma(1+\frac{AL+B+n-2\ell n}{\ell}-\frac{s}{\ell})}{\Gamma(1+\frac{AL+B+n-\ell n}{\ell}-\frac{s}{\ell})\Gamma(-AL-B-a'+\ell-n+s)}ds,$$
$$(\ell, A \neq 0 \text{ and } n, L \in N_0). \quad (25)$$

The contour integral in (25) *can be expressed in terms of Fox's* H*-function* (*see Srivastava et. al.* [4, ch.2]). *Thus the solution of the integral equation* (21) *can be written as*

$$f(x) = \frac{(-1)^n t^{-(n+a'-\ell+AL+B)}}{\ell^n A^L} x^{-AL-B} \int_0^\infty y^{-1}(y^{A+1}D_y)^L \{y^B g(y)\}$$
$$H_{2,3}^{1,1}\left[\frac{xt}{y}\middle| \begin{matrix} (1+L,\frac{1}{A}),(\frac{\ell+AL+B+n-\ell n}{\ell},\frac{1}{\ell}),(\cdots\cdots) \\ (\frac{\ell+AL+B+n-2\ell n}{\ell},\frac{1}{\ell}),(1,\frac{1}{A}),(1+AL+B+a'+n-\ell,1) \end{matrix}\right]dy. \quad (26)$$

If we take $a=1$, $b=0$, $p=1$, $\beta=0$, $\xi=0$, $\lambda=0$ *and replace* α *by* $(\alpha+kn)$ *and* ν *by* $-k$ *in* (4), *Theorem reduces to a result given by Srivastava* [7] *underless stringent conditioins. Also, for* $a=1$, $b=0$, $p=1$, $B=0$, $\xi=0$, $\lambda=\nu=0$, $k=A=-1$ *in* (5), *Theorem seems to correspond to result given by Lala and Shrivastava* [7].

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