

HODGE THEORY AND COHOMOLOGY WITH COMPACT SUPPORTS

BY

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Abstract. The main result is an isomorphism between the de Rham cohomology with compact supports of $M$ and the kernel of the Hodge–Witten–Bismut Laplacian $\Delta_\mu$ associated to a measure $d\mu$ which has sufficiently rapid growth at infinity on $M$. The isomorphism applies when the space of $C^\infty$ vectors for $\Delta_\mu$ satisfies an “extension by zero” property. This property is proved for manifolds with cylindrical ends possessing Gaussian growth measures. The results of this paper extend to the case of differential forms on $M$ with values in a flat Hermitian vector bundle.

1. Introduction

Let $M$ be a real $C^\infty$ manifold without boundary. We say $M$ is topologically tame if there exists a compact manifold with boundary $N$ such that $M$ is diffeomorphic to the interior of $N$. For instance, vector bundles over compact manifolds are topologically tame. A surface of infinite genus is not topologically tame.

Suppose $(M, g)$ is topologically tame and complete. Let $d\mu = e^{2h} dx$ be a measure on $M$ with $h \in C^\infty(M)$ and $dx$ the Riemann–Lebesgue measure. Let $L^2_\mu \Omega$ be the Hilbert space of differential forms square–integrable with respect to $d\mu$. If $d$ is the exterior derivative on differential forms and $\delta_\mu$ its adjoint in $L^2_\mu \Omega$ then (see below) $\Delta_\mu = d\delta_\mu + \delta_\mu d$ is essentially self-adjoint and nonnegative on the space $\Omega_c(M)$ of smooth forms with compact support.
Now, if $\Delta_\mu$ has spectral gap (see below) then there is an orthogonal decomposition

$$L^2_\mu \Omega = \ker \Delta_\mu \oplus \text{im } d \oplus \text{im } \delta_\mu$$

(“ker” = kernel and “im” = image) which yields a specific Hodge isomorphism

$$(*) \quad \frac{\ker d|_{L^2_\mu \Omega}}{\text{im } d|_{L^2_\mu \Omega}} \cong \ker \Delta_\mu.$$

The cohomology on the left involves the growth or decay constraints imposed by the $L^2_\mu \Omega$ structure. It is not invariant under diffeomorphisms—in this sense it is not topological.

We seek conditions on $(M, g)$ and $d\mu$ under which the de Rham cohomology

$$H_{\text{de R}}(M) = \ker d|_{\Omega(M)} / \text{im } d|_{\Omega(M)}$$

is isomorphic to the weighted spaces in $(*)$. Let $d\mu^- = e^{-2h} dx$ for the same $h \in C^\infty(M)$ used to define $d\mu$. In the special cases of Euclidean space (see below) and of vector bundles with certain prescribed metrics ([4, [20]), it is known that if $d\mu^- = e^{-Cr^2} dx$ for $C > 0$ and for $r$ the distance from the origin (resp. the compact base manifold) then the topological Hodge-de Rham isomorphism holds:

$$H_{\text{de R}}(M) \cong \ker \Delta_{\mu^-}.$$

The same techniques also show

$$(**) \quad H_{c,\text{de R}}(M) \cong \ker \Delta_{\mu^+}$$

for $d\mu = d\mu^+ = e^{+Cr^2} dx$ by Poincaré duality (Proposition 6). Here $H_{c,\text{de R}}(M)$ is the cohomology with compact supports.

We show that $(**)$ can be established in more general circumstances by checking a condition on the $C^\infty$-vectors for $\Delta_\mu$ (in the operator sense defined below, cf. [22]). The use of this condition is the main new technique of the current paper.

Note that if $M$ is topologically tame then we can find an open submanifold $M_0 \subset M$ diffeomorphic to $M$ and such that the closure of $M_0$ is a compact manifold with boundary—see Section 3.

**Main Technique.** (Theorem 13) Let $\Psi : M \to M_0$ be a diffeomorphism for $M_0 \subset M$ open. Suppose that for all $\omega \in C^\infty(\Delta_\mu) = \bigcap_{k \geq 1} D_{(\Delta_\mu)^k}$ we can show
that \((\Psi^{-1})^* \omega \in \Omega(M_0)\) has smooth extension by zero to \(M\). (We call this condition “Property EZ”.)
Then there exists a surjection \(j : \ker \Delta_\mu \to H_{c,de R}(M)\).

Observe that any nonzero \(\omega \in \ker \Delta_\mu\) does not have compact support since \(\Delta_\mu\) is elliptic. However, the eigenforms of \(\Delta_\mu\) are in \(C^\infty(\Delta_\mu)\), and they decay very rapidly if \(d\mu = e^{Cr^2} \, dx\), \(C > 0\), and \(r\) is a distance function, for instance.

For manifolds with cylindrical ends and if \(d\mu\) is of the form \(e^{Cr^2} \, dx\) we show \(\ker \Delta_\mu\) is finite dimensional. Then we show \(H_{c,de R}(M)\) surjects onto \(\ker \Delta_\mu\) by a straightforward argument. Thus:

**Main Application.** (Theorem 15) Consider the case of \((M;g)\) such that the ends of \(M\) are cylinders and of a measure \(d\mu\) with specific rapid growth on the ends of \(M\): \(d\mu = e^{Cr^2} \, dx\), for \(C > 0\). By applying the above technique we prove that with these choices of \(g\) and \(d\mu\) on \(M\), the \(\Delta_\mu\)-harmonic forms represent the de Rham cohomology with compact supports:

\[
\ker \Delta_\mu^p \cong H_{c,de R}^p(M), \quad p = 0, \ldots, n.
\]

By Poincaré duality (Proposition 6) we conclude \(\ker \Delta_\mu^- \cong H_{de R}^n(M)\) under the same hypotheses.

Similar results will hold for any Riemannian manifold \((M;g)\) and measure \(d\mu\) for which we can show property EZ and the finite dimensionality of \(\ker \Delta_\mu\).

The main technique above is compatible with the heuristic that if \(d\mu\) has sufficiently rapid growth then the elements of \(L^2_\mu \Omega\) are “practically” compactly-supported and thus \((**)\) should hold since \((*)\) holds. Readers familiar with Gaussian and other decaying measures (e.g. heat kernel measures) should note that forms in \(L^2_\mu \Omega\) decay, and thus extension by zero is conceivable (though smoothness must be addressed). By contrast, forms in \(L^2_{\mu^-} \Omega\), where \(d\mu^-\) is a decaying measure, typically grow at \(\infty\). The Laplacian \(\Delta_{\mu^-}\) of Example 5 has eigenforms which are Hermite polynomials, for instance. See Proposition 6 for the duality between \(\Delta_{\mu^+}\) and \(\Delta_{\mu^-}\).

The remainder of the introduction defines \(\Delta_\mu\) and also recalls the Hodge theory of unweighted \(\Delta\) in the case of \((M,g)\) with cylindrical ends. Comparison then shows the role of densities \(e^{\pm r^2}\) in the current paper.

The metric \(g\) determines an inner product \((\cdot, \cdot)_x\) on the tangent space \(T_xM\) by definition, which induces an inner product \((\cdot, \cdot)_x\) on \(\wedge T^*_x M\), the space of pointwise
values of differential forms. The metric also determines a measure \(dx\), by the coordinate formula
\[
dx = \sqrt{|\det g_{ij}|} dx_1 \ldots dx_n.
\]
One defines an unweighted inner product on the space of smooth \((C^\infty)\) compactly–supported differential forms \(\Omega_c(M)\) by the formula
\[
\langle \omega, \nu \rangle = \int_M (\omega \wedge \nu) dx = \int_M \omega \wedge * \nu. \quad (\text{See [24], for instance, for details.})
\]
We will assume for convenience that \(M\) is oriented.

We weight the inner product. If \(h \in C^\infty(M)\) then define
\[
d\mu = e^{2h} dx \quad \text{and} \quad \langle \omega, \nu \rangle_\mu = \int_M (\omega, \nu) e^{2h} d\mu.
\]
Let \(L^2_\mu \Omega\) be the completion of \(\Omega_c(M)\) with respect to \(\langle \cdot, \cdot \rangle_\mu\).

The exterior derivative \(d\) has formal adjoint \(\delta_\mu : \langle d\omega, \nu \rangle_\mu = \langle \omega, \delta_\mu \nu \rangle\) for \(\omega, \nu \in \Omega_c\). In fact, the formula \(\delta_\mu = e^{-2h} \delta e^{2h}\) holds, where \(\delta = (-1)^n(p+1)+1 * d*\) acting on \(\Omega^p\) is the formal adjoint of \(d\). Let \(\Delta_\mu = d\delta_\mu + \delta_\mu d\). The following theorem is well–known.

**Theorem 1.** The operator \(\Delta_\mu\) defined on \(\Omega_c\) is densely–defined and symmetric with respect to \(\langle \cdot, \cdot \rangle_\mu\), and is a second–order elliptic operator which maps \(\Omega^p_c\) to itself. If \((M,g)\) is complete, we can extend \(\Delta_\mu\) to a unique nonnegative (self–adjoint) operator in \(L^2_\mu \Omega\) ([14], [9] for the case where \(d\mu = dx\), [7] for the weighted case).

If \(M^n\) is in fact compact, then \(\Delta_\mu\) has compact resolvent (discrete spectrum).

The Hodge decomposition of the smooth forms \(\Omega^p(M)\), \(p = 0, \ldots, n\), follows:
\[
\Omega^p(M) = \ker \Delta_\mu^p \oplus \text{im} \delta^{p-1} \oplus \text{im} \delta^{p+1}, \quad (M \text{ compact})
\]
from which it follows that \(H^p_{de R}(M) \cong \ker \Delta_\mu^p\).

The weighting \(d\mu = e^{2h} dx\) on \(L^2\) spaces of forms has been considered in connection with the Witten treatment of the Morse inequalities ([25]). Actually, our \(\Delta_\mu\) on \(L^2_\mu \Omega\) is unitarily–equivalent to the Laplacian usually considered in that context, as explicitly noted in [4]. Let \(L^2 \Omega = L^2_0 \Omega\). There is the following unitary equivalence:

\[
\begin{align*}
L^2_\mu \Omega & \quad \xrightarrow{U} \quad L^2 \Omega \\
\omega & \quad \mapsto \quad e^h \omega \\
d & \quad \mapsto \quad d_h = U d U^{-1} = e^h d e^{-h} \\
\Delta_\mu = d\delta_\mu + \delta_\mu d & \quad \mapsto \quad \Delta_h = U \Delta_\mu U^{-1} = \Delta + |dh|^2 + A_h,
\end{align*}
\]
where $A_h$ is a certain zeroth order symmetric operator on $L^2\Omega$ depending linearly on the second derivatives of $h$.

We want to consider noncompact topologically tame manifolds $M$. On such $M$ we can explicitly construct metric and measure which yield nice spectral properties.

**Theorem 2.** ([7]) Let $M$ be a $C^\infty$, orientable, connected, and topologically tame manifold. There exists a metric $g$ (under which the ends of $M$ are cylinders near infinity) and for each $C \neq 0$ a measure $d\mu$ with smooth density with respect to $dx$ (which behaves precisely as $e^{Cr^2}$ near infinity on the ends) such that

(i) $(M, g)$ is complete and

(ii) $\Delta_\mu$ in $L^2\Omega_\mu$ has compact resolvent (= discrete spectrum).

If $\Delta_\mu$ on $L^2\Omega_\mu$ has compact resolvent or merely spectral gap, then a certain Hodge decomposition, sometimes called a strong Hodge decomposition, follows.

**Proposition 3.** Suppose $(M^n, g)$ is complete. If $\Delta_\mu$ has spectral gap (i.e. $\sigma(\Delta_\mu) \setminus \{0\} \subseteq [\epsilon, \infty)$ for some $\epsilon > 0$) or if $\Delta_\mu$ is Fredholm and $\text{im} d$ is closed, then

$$L^2\Omega_\mu^p = \ker \Delta_\mu^p \oplus \text{im} d^{p-1} \oplus \text{im} \delta^{p+1}_\mu, \quad p = 0, \ldots, n.$$ 

The conclusion of Proposition 3 is not strong in the sense that there is a priori no relation between $\ker \Delta_\mu$ and either $H_{\text{de R}}(M)$ or $H_{\text{c,de R}}(M)$ in the noncompact case. By contrast, in the case covered by Theorem 2 we will show that if $C > 0$ then $\ker \Delta_\mu \cong H_{\text{c,de R}}(M)$ and if $C < 0$ then $\ker \Delta_\mu \cong H_{\text{de R}}(M)$.

The widely studied case $d\mu = dx$ provides plenty of examples showing that $\ker \Delta$ and $H_{\text{de R}}(M)$ at best stand in a very complicated relation to each other. (We write $\Delta$ for $\Delta_{\text{de R}}$.) An example is that if $M$ has the additional structure of a Riemannian cover of a compact quotient manifold $M/\Gamma$, then the space $\ker \Delta$ is determined by the topology of $M/\Gamma$ and by the group $\Gamma$, in the manner first described by Atiyah [2]. It should be clear that usually these noncompact $M$, with compact quotient $M/\Gamma$, are not topologically tame. Conversely, many topologically tame noncompact $M$ admit no metric under which there exists a group of isometries $\Gamma$ with compact quotient $M/\Gamma$. 
Another example is the following theorem, which describes the unweighted $L^2$ cohomology in the case of manifolds with cylindrical ends. Contrast this result with the weighted case in “Main Application”.

Theorem 4. ([3], Proposition 4.9) Assume $(M, g)$ is complete, is topologically tame and has cylindrical ends. Let $i : \Omega_c(M) \to \Omega(M)$ be the inclusion map. The space $\ker \Delta$ (of $L^2_{dx} \Omega$ harmonic forms) is isomorphic to the image of the induced map $i$ from $H_{c, de R}(M)$ into $H_{de R}(M)$.

Although this result establishes a connection between the space of $L^2_{dx}$ harmonic forms and the (topological) cohomologies, it shows that $\ker \Delta$ must vanish in circumstances where there is much cohomology to be represented. For instance, $\ker \Delta^p$ vanishes for all $0 \leq p \leq n$ if $M^n$ is not compact and either $H^p_{de R}(M) = 0$ for $1 \leq p \leq n/2$ or $H^p_{de R}(M) = 0$ for $n/2 \leq p \leq n - 1$. Recall that $\ker \Delta^p \cong \ker \Delta^{n-p}$ for any $(M, g)$ and that $H^p_{de R}(M) = 0$ if $M$ is not compact.

Theorem 4 and other results in [3] have motivated a family of results on manifolds with cylindrical ends. R. Melrose and coworkers (specifically, [19]) address the situation of a compact manifold with boundary, and a certain prescribed metric near the boundary which they call an “exact $b$-metric”. Such a metric is cylindrical near the boundary. Section 6.4 of [19] includes a Hodge theorem which relates to our Theorem 15 after mapping the manifold with boundary and exact $b$–metric to a noncompact manifold with cylindrical ends. However, the technique given in Section 3 of this paper applies to manifolds with noncylindrical ends, and directly addresses the complete noncompact manifold itself. Thus Theorem 15 at least represents an extension of the ideas in [19].

We are motivated to consider measures with particular decay rates by the following example, based on well-known spectral properties of the quantum harmonic oscillator.

Example 5. Let $d\mu^+ = e^{+|x|^2} dx$, $d\mu^- = e^{-|x|^2} dx$ on Euclidean space $\mathbb{R}^n$. Then $\Delta^+ \mu$ and $\Delta^- \mu$ have compact resolvent and

$$\ker \Delta^p_{\mu^-} \cong \begin{cases} \mathbb{R}, & p = 0 \\ 0, & p > 0, \end{cases} \quad \ker \Delta^p_{\mu^+} \cong \begin{cases} \mathbb{R}, & p < n \\ 0, & p = n. \end{cases}$$
Note that

\[ H^p_{\text{de R}}(\mathbb{R}^n) \cong \begin{cases} \mathbb{R}, & p = 0 \\ 0, & p > 0 \end{cases}, \quad H^p_{\text{c,de R}}(\mathbb{R}^n) \cong \begin{cases} 0, & p < n \\ \mathbb{R}, & p = n. \end{cases} \]

In general, if \( d\mu^+ = e^{+2h}dx \) and \( d\mu^- = e^{-2h}dx \) on \( M \), then from (1)

\[ \Delta^+_h = U_+ \Delta_+, U_+^{-1} = \Delta + |dh|^2 + A_h, \]

\[ \Delta^-_h = U_- \Delta_-, U_-^{-1} = \Delta + |dh|^2 - A_h, \]

are the unitarily equivalent versions of \( \Delta_+ \), \( \Delta_- \) respectively, both acting on (unweighted) \( L^2\Omega \). Thus the difference between \( \Delta_+ \) and \( \Delta_- \) is determined only by the sign of the potential term \( A \) (an endomorphism on \( \wedge T^*M \)), once we push \( \Delta_+ \), \( \Delta_- \) back into the same space \( L^2\Omega \). On the other hand, note that \( L^2_{\mu_+}\Omega \) and \( L^2_{\mu_-}\Omega \) contain forms of (in general) very different growth/decay at infinity.

In the rest of this paper, we will drop “de R” from our notation.

Now, if \( \ker \Delta^p_\mu \) represents \( H^p(M) \) for all \( 0 \leq p \leq n \) for a given noncompact \( M \), then it will not also represent \( H^p_c(M) \). However, as suggested by Example 4, measures with inverse densities give a desirable relationship.

**Proposition 6. Poincaré duality.** If \( M^n \) is orientable, then

\[ H^p(M) \cong (H^{n-p}_c(M))^*. \]

**Weighted Poincaré duality.** If \( (M^n, g) \) is orientable, and \( d\mu^+ = e^{+2h}dx \), \( d\mu^- = e^{-2h}dx \), and if \( *\mu = e^{2h}*, \) then

\[ *\mu \Delta^p_\mu = \Delta^{n-p}_\mu *\mu \]

and thus \( \ker \Delta^p_- \cong \ker \Delta^{n-p}_+ \).

**Proof.** For Poincaré duality, see [5]. For this weighted version, see [12].

It follows that if we find a measure \( d\mu = e^{2h}dx \) for which \( \ker \Delta^p_\mu \cong H^p(M) \), then defining \( d\mu^- = e^{-2h}dx \), we get (for free!) \( \ker \Delta^p_- \cong H^p(M) \). And vice versa of course. Theorem 15 shows that this is the situation that holds for a topologically tame manifold, if we have chosen \( g \) and \( d\mu \) as in Theorem 2.
The methods of this paper are closely related to those in [20], where, in particular, the isomorphism between the $L^2$ cohomology of a vector bundle $E$, corresponding to a density $e^{C|x|^2}$ for $C > 0$, and the compact cohomology $H_c(E)$ was proven, and then used in a proof of the Morse–Bott inequalities.

However, the particular form of Definitions 7, 9 and 11 and of Theorem 13 are motivated by the following situation. If $(M, g)$ is complete and has Ricci curvature bounded below, then the usual Laplace–Beltrami operator $-\frac{1}{2}\Delta$ on $M$ has a unique heat kernel $\rho$. On Euclidean space $\mathbb{R}^n$, $\rho_t(x, y) = (2\pi t)^{-n/2} \exp(-|x - y|^2/2t)$. On $M$ with Ricci bounded below, if $t > 0$ and $x_0 \in M$ then $d\mu^\rho \equiv \rho_t(x_0, x)dx$ is a probability measure on $M$. It is conjectured in [7] that $\ker \Delta_{\mu^\rho} \cong H^p(M)$ for this heat kernel measure $d\mu^\rho$, and certain evidence is given there. It is hoped that the methods of this paper will apply to $d\mu^{\rho, +} = (\rho_t(x_0, x))^{-1} dx$, in which case the conjecture would hold true. See [15] for a proof of the $n$–forms case, where $n = \dim M$, of the isomorphism $\ker \Delta_{\mu^\rho} \cong H^n_{de R}(M)$. In particular, it is proved there that $\int_M \rho_t^{-1} dx = \infty$ if $M$ is not compact and the Ricci curvature is bounded below.

Recent work of Z. Ahmed and D. Stroock [1] contrasts with the results of this paper and of others which trace back to [25]. They prove that if $d\mu$ is a measure with such strong decay that the semigroup $e^{t\Delta_{\mu}}$ takes $L^2_{\mu}$ into $L^\infty$ (“ultraccontractivity”) then $\ker \Delta_{\mu} \cong H_{de R}(M)$. Among other things, this contraction property implies that the eigenforms of $\Delta_{\mu}$ are actually bounded. Since this is not true for $d\mu^\pm = e^{-|x|^2} dx$ on Euclidean space $\mathbb{R}^n$ (cf. [11]), the Ahmed–Stroock result will not apply to the quadratic growth/decay case of Theorem 15, for instance. Their result applies to $d\mu = \exp(-|x|^{2+\epsilon}) dx$ ($\epsilon > 0$) for $\mathbb{R}^n$ and some other noncompact manifolds.

The results of this paper extend to the case of differential forms with values in a flat Hermitian vector bundle, provided that this bundle satisfies some natural conditions at infinity. We give the conditions in Section 6. This extension has applications to the index theory of manifolds with cylindrical ends ([18], [23]).

The technique and results of this paper appeared in preprint form in 1998 ([8]).

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2. $C^\infty_\mu$: A Space of Forms of Very Rapid Decay

**Definition 7.** Assume $(M, g)$ is complete, so $\Delta_\mu$ is self-adjoint. Define

$$C^\infty_\mu \equiv C^\infty(\Delta_\mu) = \bigcap_{k \geq 1} D(\Delta_\mu)^k.$$ 

Clearly $\Omega_c(M) \subseteq C^\infty_\mu$, so $C^\infty_\mu$ is never empty. If $\Delta_\mu$ has eigenforms then they are evidently in $C^\infty_\mu$. The ellipticity of $\Delta_\mu$ implies that $C^\infty_\mu$ forms are smooth. If $M$ is compact, then $C^\infty_\mu = \Omega(M)$. If $M$ is not compact, then $C^\infty_\mu$ includes constraints on growth at infinity, and in fact $\Omega_c(M) \subset C^\infty_\mu \subset \Omega(M)$ are both proper inclusions.

It is appropriate to think of $C^\infty_\mu$ as some kind of generalization to forms of the Schwartz space of functions, with growth constraints determined by $d\mu$. More accurately, $e^b C^\infty_\mu = C^\infty(\Delta_b)$ generalizes Schwartz space—see Section 5.

Consider the example:

**Example 8.** Let $M = \mathbb{R}^n$ with the Euclidean metric, and let $d\mu = e^{+|x|^2} dx$. The unitarily equivalent operator $\Delta_b$ on $L^2 \Omega$ corresponding to $\Delta_\mu$ on $L^2_\mu \Omega$ is a harmonic oscillator Hamiltonian $\Delta_b^p = \Delta^p + |x|^2 + n - 2p$ (on $L^2 \Omega^p$).

Considering only functions, it is well known that $C^\infty(\Delta_b^0)$ corresponds to the Schwartz space $S(\mathbb{R}^n)$. The family of seminorms $\{\|f\|_k = \|(\Delta_b^0)^k f\|_{L^2}\}$ is equivalent to the usual family of seminorms $\{\|f\|_{\alpha, \beta} = \|x^\alpha D^\beta f\|_{L^2}\}$ ([22], Appendix to V.3). Thus $f \in C^\infty(\Delta_b^0)$ iff $\|f\|_k < \infty$ for all $k$ iff $f \in S(\mathbb{R}^n)$. In fact, the zero form $g \in C^\infty_\mu$ iff $g = e^{-|x|^2/2} f$ for $f \in S(\mathbb{R}^n)$. It follows that forms in $C^\infty_\mu$ have “very rapid” decay.

**Definition 9.** $H_\mu$ is the cohomology of $(C^\infty_\mu, d)$:

$$H^p_\mu = \frac{\ker d|_{C^\infty_\mu}}{\text{im } d|_{C^\infty_\mu}}, \quad p = 0, \ldots, n.$$ 

The closure of $\text{im } d|_{C^\infty_\mu}$ is not taken in our definition of $H_\mu$, but $H_\mu \cong \ker \Delta_\mu$ if $\Delta_\mu$ has a spectral gap:

**Corollary 10.** The Hodge decomposition of Proposition 3 can be smoothed: If $\Delta_\mu$ has a spectral gap, then

$$C^\infty_\mu = \ker \Delta_\mu \oplus \text{im } d|_{C^\infty_\mu} \oplus \text{im } \delta_\mu|_{C^\infty_\mu}.$$
so $\ker \Delta_\mu \cong H_\mu$.


Suppose $M^n$ is topologically tame and is equipped with a Riemannian metric $g$ making $(M, g)$ a complete metric space. Then we can think of $M$ as including a precompact part carrying all the topology, and a finite list of infinite–length ends which are topological products. In fact, suppose $N$ is a compact manifold with boundary with $\Phi : \text{int}N \to M$ a diffeomorphism. Let $U \subset N$ be an open neighborhood of $\partial N$ which is small enough so that $U \cong \partial N \times (0, \infty)$. Decompose $U$ into finitely–many connected pieces, so there exists diffeomorphism $A : \bigsqcup Q_i \times (0, \infty) \to U$, with each $Q_i$ compact without boundary. The boundary $\partial N$ is identified with points $(q, \infty)$ in this representation of $U$. Let $E_i = \Phi(A((Q_i \times (0, \infty)))$ (an end of $M$) and let $M_0 = M \setminus \Phi (A \left( \bigsqcup Q_i \times [1, \infty)) \right.)$. Then $M_0 \cap E_i \cong Q_i \times (0, 1)$ and $M_0$ is diffeomorphic to $M$.

**Definition 11.** Property EZ. Let $M$ be topologically tame, assume $(M, g)$ is complete, and suppose a measure $d \mu = e^{2h} dx$ is given. Consider $M_0 \subset M$ an open submanifold of $M$ with closure a compact manifold with boundary such that there exists a diffeomorphism $\Psi : M \to M_0$. We say $d \mu$ has Property EZ if there exist $M_0$ and $\Psi$ so that

every $\omega \in (\Psi^{-1})^* C^\infty_\mu$ has a smooth extension by zero to all of $M$.

Note $H_\mu(M)$ is finite–dimensional if $M$ is topologically tame.

**Lemma 12.** Let $M$ be a topologically tame manifold with $M_0 \subset M$ open, diffeomorphic to $M$, and with closure a compact manifold with boundary. Let $i : \Omega_c(M_0) \to \Omega_c(M)$ be the map which extends forms by zero. Then $i$ induces an isomorphism $i : H_c(M_0) \to H_c(M)$.

**Proof.** We construct a family $\Psi_\epsilon$, $0 < \epsilon < 1$, of $M \to M_0$ diffeomorphisms which fix “large portions” of $M_0$. First, note that $\Psi(M_0)$ is diffeomorphic to $M$, since $\Psi|_{M_0} \circ \Psi$ is a diffeomorphism. Thus $M \setminus \Psi(M_0)$ is diffeomorphic to $\partial M_0$ times an interval. Let $\alpha : \partial M_0 \times [-1, \infty) \to M \setminus \Psi(M_0)$ be such a diffeomorphism,
chosen so that $\alpha(\partial M_0 \times [-1, 0)) = M_0 \setminus \Psi(M_0)$ and $\alpha(\partial M_0 \times [0, \infty)) = M \setminus M_0$, so $\alpha(\partial M_0 \times \{0\}) = \partial M_0$. Let $M_\varepsilon = M_0 \setminus \alpha(\partial M_0 \times (-\varepsilon, \infty))$ for $0 < \varepsilon < 1$, and also define $\beta_\varepsilon : [-1, \infty) \to [-1, 0)$ so that $\beta_\varepsilon(\xi) = \xi$ for $-1 \leq \xi \leq -\varepsilon$, and $\beta_\varepsilon$ is smooth and monotone increasing with $\lim_{\xi \to \infty} \beta_\varepsilon(\xi) = 0$.

Define $\Psi_\varepsilon(x) = x$ for $x \in \Psi(M_0)$. For $x \in M \setminus \Psi(M_0)$, let $(y, \xi) = \alpha^{-1}(x)$, $y \in \partial M_0$, $\xi \in [-1, \infty)$. Define $\Psi_\varepsilon(x) = \alpha(y, \beta_\varepsilon(\xi))$. Then $\Psi_\varepsilon$ fixes $M_\varepsilon$ and is a diffeomorphism $M \to M_0$ for each $0 < \varepsilon < 1$. And $\Psi_\varepsilon$ induces $\Psi_\varepsilon^*$ on forms and an isomorphism $\Psi_\varepsilon^*$ on cohomology.

If $\alpha = [\omega] \in H_\varepsilon(M_0)$, $\alpha \neq 0$, we will show $i\alpha \neq 0$ in $H_\varepsilon(M)$. This suffices since $H_\varepsilon(M_0)$, $H_\varepsilon(M)$ are finite-dimensional. Since $\omega$ has compact support in $M_0$, there exists $\varepsilon > 0$ such that $\text{supp } \omega \subset M_\varepsilon$. Then $\Psi_\varepsilon^* \omega = i \omega$. But $i \alpha = [i\omega] = [\Psi_\varepsilon^* \omega] = \Psi_\varepsilon^* \alpha \neq 0$ in $H_\varepsilon(M)$ since $\Psi_\varepsilon^*$ is an isomorphism.

**Theorem 13.** Suppose $M^n$ is topologically tame, assume $(M, g)$ is complete, and suppose the measure $d\mu = e^{2h}dx$ is given. If $d\mu$ has property EZ, then the map $j : C^\infty_\varepsilon \to \Omega_\varepsilon(M)$ which extends $\omega \in (\Psi^{-1})^* C^\infty_\varepsilon$ by zero induces a surjection

$$j : H_\mu \to H_\varepsilon(M).$$

**Proof.** Let $M_0$, $\Psi$ be as in the definition of property EZ. As in the proof of Lemma 12, for $\varepsilon > 0$ define $M_\varepsilon \subset M_0$ and modify $\Psi$ so that it fixes $M_\varepsilon$—clearly property EZ is independent of the behavior of $\Psi$ on any compact set. Let $k_\varepsilon : \Omega_\varepsilon(M_\varepsilon) \to C^\infty_\varepsilon$ be extension by zero. Then $i_\varepsilon$, extension by zero from $M_\varepsilon$ to $M$, factors through $j$ : $i_\varepsilon = j \circ k_\varepsilon$.

By the lemma $i_\varepsilon$ is an isomorphism in cohomology. Both $k_\varepsilon$ and $j$ induce maps on cohomology. It follows that $k_\varepsilon : H_\varepsilon(M_\varepsilon) \to H_\mu$ is an injection and $j : H_\mu \to H_\varepsilon(M)$ is a surjection.

Recalling Poincaré duality, we see that if $d\mu$ on $(M^n, g)$ has property EZ then

$$\dim H^p_\mu \geq \dim H^p_\varepsilon(M) = \dim H^{n-p}(M) \equiv b_{n-p}.$$ 

We do not exclude the possibility that $\dim H^p_\mu = \infty$, nor do we assume that $H^p_\mu$ has a Hilbert space structure. Also, it is not assumed that $\Delta_\mu$ has a spectral gap.
There is an available map going the other direction from \( f \). Consider the orthogonal projection \( \Pi : L^2_\mu \Omega \rightarrow \ker \Delta_\mu \). This map clearly restricts to \( \Pi : \Omega_c(M) \rightarrow \ker \Delta_\mu \). Note \( \langle d\nu, \varphi \rangle_\mu = \langle \nu, \delta_\mu \varphi \rangle_\mu = 0 \) for all \( \nu \in \Omega_c(M) \) and \( \varphi \in \ker \Delta_\mu \). This shows that \( \Pi \) is well-defined as a map on compactly-supported cohomology:

**Definition 14.** Define

\[
\Pi : H_c(M) \rightarrow \ker \Delta_\mu \quad \text{by} \quad [\omega] \mapsto \Pi(\omega).
\]

If \( d\mu \) has property EZ, then the following method may also apply. First, the natural map \( \eta : \ker \Delta_\mu \rightarrow H_\mu \), given by \( \omega \mapsto [\omega] \) is always injective if \( M \) is complete, since if \( \omega = d\nu \) for \( \nu \in C^\infty_\mu \) then \( \langle \omega, \varphi \rangle_\mu = \langle \nu, \delta_\mu \varphi \rangle_\mu = 0 \) for all \( \varphi \in \ker \Delta_\mu \). If \( d\mu \) has property EZ, and if the composition

\[
\ker \Delta_\mu \xrightarrow{\eta} H_\mu \xrightarrow{j} H_c(M) \xrightarrow{\Pi} \ker \Delta_\mu
\]

is surjective, then \( \Pi \) is surjective. This reduces to checking whether an explicit endomorphism \( \Pi \circ j \circ \eta \) on \( \ker \Delta_\mu \), depending only on \( \Psi \) and \( d\mu \), is surjective.

Whether \( \Pi \) is surjective or not is \textit{a priori} independent of the existence of a spectral gap for \( \Delta_\mu \).

In Section 4 we will see that both \( j \) and \( \Pi \) are surjective in the case described by Theorem 2 if \( c > 0 \), and thus that \( H_\mu \cong H_c(M) \). That is, we prove that if: (i) \( M \) is topologically tame, (ii) the ends of \( (M, g) \) are cylinders, and (iii) \( d\mu \) is a measure with density \( \exp(\lambda r^2) \), \( c > 0 \) on the ends, then \( d\mu \) has property EZ. It follows that \( j \) is surjective, and we prove \( \Pi \) is surjective by the method described above.

### 4. Cylindrical Ends, Gaussian Growth Measure

Let us return to the setting of Theorem 2. Let \( M \) be topologically tame, and define \( g \) as a metric under which \( (M, g) \) is complete and the ends of \( M \) are cylinders. Define \( d\mu \) as in Theorem 2 as well, with \( c > 0 \): \( d\mu = e^{cr^2} dx \). (We will more explicitly define \( g \) and \( d\mu \) in a moment.) In this section we prove that \( d\mu \) on \( M \) has property EZ, and in fact we prove that both \( j \) and \( \Pi \) of the previous section
are surjections. As we will see, the proof of property EZ ultimately depends on a fundamental estimate for $C_1^\infty$ (actually $C_h^\infty$) forms on $(M, g)$. The estimate is found in Theorem 16 and Corollary 17 of Section 5.

**Theorem 15.** Let $c > 0$. Let $M, g$, $d\mu = e^{c(x^1)^2} dx$ be as in Theorem 2. Then in addition to (i) and (ii), there exist surjections $\Pi$ and $j$ such that

$$H_c(M) \xrightarrow{\Pi} \ker \Delta_\mu \cong H_\mu \xrightarrow{j} H_c(M).$$

It follows that

$$H_\mu \cong H_c(M).$$

**Proof that $j$ is surjective.** We show that $d\mu$ has property EZ, so that by Theorem 13, $j : H_\mu \to H_c(M)$ is a surjection.

Since $M$ is topologically tame, $M = M_0 \cup (\bigcup_i E_i)$, with $M_0$ relatively compact, $E_i \cong (0, \infty) \times Q_i$, and $Q_i$ compact without boundary. Let $g$ be a product metric on each $E_i$ and otherwise choose $g$ arbitrarily.

Fix an end $E = (0, \infty) \times Q$, and let $Q = \bigcup_j U_j$ be a finite cover of $Q$ by coordinate neighborhoods. Let $(x^1, x^2, \ldots, x^n) = (x^1, \bar{x})$ be coordinates on $(0, \infty) \times U_j$.

Define $d\mu = e^{c(x^1)^2} dx$ on $(0, \infty) \times U_j$, therefore on $E$. Extend $d\mu$ to all of $M$ in the obvious way, so $d\mu = e^{+2h} dx$ on $M$, where $h \in C^\infty(M)$ and $h = c(x^1)^2$ on each end, for the given $c > 0$.

Define $\Psi$ on $E$ as follows. Let $M_1 = M_0 \cup (\bigcup_i (0, 1) \times Q_i)$, with coordinates $(s, \bar{y})$ on $(0, 1) \times U_j$. We define $\Psi^{-1} : M_1 \to M$ by defining $\Psi^{-1} = \text{id}$ on $M_0$ and by defining $\Psi^{-1} : (0, 1) \times U_j \to (0, \infty) \times U_j$ as follows: $\Psi^{-1}(s, \bar{y}) = ((1 - s)^{-1}, \bar{y})$ if $1/2 \leq s < 1$, and smoothly and monotonically interpolate $\Psi^{-1}$ for $0 \leq s \leq 1/2$. Clearly $\Psi^{-1}$ is a diffeomorphism and $M_1 = \Psi(M)$ has compact closure.

To prove that $d\mu$ has property EZ, it suffices to prove that if $\omega \in C_\mu^\infty$ and supp $\omega \subset E$, then $(\Psi^{-1})^* \omega$ has smooth extension by zero to $E$. Note that if $\varphi_0, \{\varphi_i\}$ is a partition of unity subordinate to $M = M_1 \cup (\bigcup_i E_i)$ then $\omega \in C_\mu^\infty$ iff $\varphi_i \omega \in C_\mu^\infty$ for each $i$.

Now we arrange to use the properties of Schwartz forms on an end $E$—see Section 5—to prove the extension by zero on $E$ for $(\Psi^{-1})^* \omega$. 
In fact, recall that $\Delta_h = e^h \Delta_m e^{-h} = \Delta + |dh|^2 + A_h$ on $L^2\Omega$ is unitarily-equivalent to $\Delta_m$ on $L^2\mu\Omega$. Here $h = \frac{c}{2}(x^1)^2$. Let $\nu \in C^\infty \cap \mathcal{D}(\Delta_m)$. Then $e^{-h}\nu \in C^\infty$. Suppose $\nu = \sum_{i \neq j} u_j dx^i \wedge dx^j + \sum_{i} u_i dx^i$ on $(0, \infty) \times U_j$. Then since $\frac{dx_i}{ds} = (1-s)^{-2}$,

$$
\omega' \equiv \left(\Psi^{-1}\right)^*(e^{-h}\nu) = e^{-\frac{c}{2}(1-s)^{-2}}(1-s)^{-2}u_f((1-s)^{-1}, \bar{y}) ds \wedge dy^j + e^{-\frac{c}{2}(1-s)^{-2}}u_f((1-s)^{-1}, \bar{y}) dy^f
$$

is in $\Omega(M_1)$.

Denote as $\frac{\partial}{\partial x^j} : \Omega^p((0, \infty) \times U_j) \to \Omega^p((0, \infty) \times U_j)$ the operator defined by

$$
\frac{\partial}{\partial x^j} \omega = \sum \frac{\partial \omega_j}{\partial x^j} dx^j
$$

Clearly, if

$$
|(x^1)^k(\partial/\partial x)^\alpha \nu(x^1, \bar{x})| \to 0 \quad \text{as} \quad x^1 \to \infty \quad (2)
$$

for every $k \geq 0$ and every multiindex $\alpha$, then it follows that

$$
|(\partial/\partial y)^\alpha \omega'(s, \bar{y})| \to 0 \quad \text{as} \quad s \to 1^- \quad (3)
$$

for every $\alpha$, where $(y^1, y^2, \ldots, y^n) = (s, \bar{y})$ are coordinates on $(0, 1) \times U_j$.

But (2) is exactly the conclusion of Corollary 17, and (3) is exactly what is needed to prove property EZ on the end $E$. It follows from Theorem 13 that $j : H_\mu \to H_{e}(M)$ is surjective.

**Proof that $\Pi$ is surjective.** Let $\{\omega^1, \ldots, \omega^N\}$ be an orthonormal basis for $\ker \Delta_m$. Note $N < \infty$ since $\Delta_m$ has compact resolvent. For each $\omega^i$, we will find $\nu \in \Omega_{\nu}(\mu)$ such that $d\nu = 0$ and $\Pi(\nu) = \omega^i$.

Choose $\epsilon = \epsilon(N) > 0$ so that any $N \times N$ real matrix $(a_{ij})$ with $|a_{ij} - 1| < \epsilon$ for all $i$ and $|a_{ij}| < \epsilon$ for $i \neq j$ is invertible. For instance, $\epsilon = N^{-1}$ suffices.

In what follows we will make a finite number of choices for $R$, using the finite-dimensionality of $\ker \Delta_m$ and the finite list of ends $E_i$, and we assume that the maximum of these $R$ is used as needed.

For any $R \geq 0$, define $\Psi_R$ analogously to $\Psi$ on a particular end $E$: $\Psi^{-1}_R$ is the identity on $\{s \leq R\} = M_R$, $\Psi^{-1}_R(s, \bar{y}) = (R + (1 - (s - R))^{-1}, \bar{y})$ for $R + 1/2 \leq s < R + 1$, and smoothly and monotonically interpolate $\Psi^{-1}$ for $R \leq s \leq R + 1/2$. Thus $\Psi_R$ is a diffeomorphism and $\Psi_R(M) = M_{R+1}$. We may assume $|d\Psi^{-1}_R| \leq 4$ on $R \leq s \leq R + 1/2$. 


Next, we use property EZ: define $j_R$ on forms $\varphi \in C^\infty_\mu$ as the extension by zero of $(\Psi_R^{-1})^* \varphi$. Let $\varphi^i = j_R \omega^i$. Each $\varphi^i \in \Omega_c(M)$ and $d\varphi^i = (\Psi_R^{-1})^* d\omega^i = 0$, so $[\varphi^i] \in H_c(M)$.

Clearly $|\langle \varphi^i, \omega^j \rangle_\mu| \leq ||\varphi^i - \omega^j||_\mu$ if $i \neq j$ and $|\langle \varphi^i, \omega^i \rangle_\mu - 1| \leq ||\varphi^i - \omega^i||_\mu$. We will show that we can choose $R \geq 0$ so that $||\varphi^i - \omega^i||_\mu < \epsilon$. It will follow that $A = (\langle \varphi^i, \omega^j \rangle_\mu)$ is invertible.

Note $\varphi^i|_{M^r} = \omega^i|_{M^r}$. Thus

$$||\varphi^i - \omega^i||^2_\mu = \int_{R<s} |\varphi^i - \omega^i|^2 d\mu \leq 2 \int_{R<s<R+1} |\varphi^i|^2 d\mu + 2 \int_{R<s} |\omega^i|^2 d\mu.$$ 

Now,

$$\int_{R<s<R+1/2} |\varphi^i|^2 d\mu \leq \int_{R<s<R+1/2} \left| \frac{\partial x_1}{\partial s} \right|^2 |\omega^i(x^1(s), \tilde{y})|^2 e^{c\gamma^2} ds d\tilde{y}$$

$$\leq 4^2 \int_{R<x^1} |\omega^i(x^1, \tilde{x})|^2 e^{c(x^1)^2} dr d\tilde{x},$$

so the essential estimate is:

$$\int_{R+1/2<s<R+1} |\varphi^i|^2 d\mu = \int_{R+1/2<s<R+1} (1 - (s - R))^{-2} |\omega^i(x^1(s), \tilde{y})|^2 e^{c\gamma^2} ds d\tilde{y}$$

$$= \int_{R+2<x^1} |\omega^i(x^1, \tilde{x})|^2 \exp(c(R+1-(x^1-R)^{-1})^2) dx^1 d\tilde{x}$$

$$\leq \int_{R+2<x^1} |\omega^i(x^1, \tilde{x})|^2 \exp(c(R+1)^2) dx^1 d\tilde{x}$$

$$\leq \int_{R+1<x^1} |\omega^i(x^1, \tilde{x})|^2 e^{c(x^1)^2} dx^1 d\tilde{x}.$$ 

It follows that

$$||\varphi^i - \omega^i||^2_\mu \leq 36 \int_{R<x^1} |\omega^i|^2 d\mu.$$ 

Choose $R \geq 0$ so that $||\varphi^i - \omega^i||_\mu < \epsilon = \epsilon(N)$ for every $i = 1, \ldots, N$. Then $A = (\langle \varphi^i, \omega^j \rangle_\mu)$ is invertible.

But

$$\Pi(\varphi^i) = \sum_j (\langle \varphi^i, \omega^j \rangle_\mu)\omega^j = A_{ij}\omega^j,$$

so if $\nu = A_{ij}^{-1} \varphi^j$ then

$$\Pi(\nu) = A_{ij}^{-1} \Pi(\varphi^j) = A_{ij}^{-1} A_{jk}\omega^k = \omega^i.$$
We have shown that $\Pi$ is surjective, as in the method described in Section 3.

5. Schwartz Forms on a Cylindrical End

In this section we show that elements of $C_h^\infty = \bigcap_{k \geq 1} D(\Delta_h)^k$ satisfy the estimates one expects from our claim that $C_h^\infty$ is a space of “Schwartz forms”. Though all of the results in this section can be extracted with sufficient effort from existing literature, we feel clarity requires a direct proof. The case of cylindrical ends is an illustration of the estimates necessary to prove “property EZ”.

Let $E = (0, \infty) \times Q$, where $Q$ is compact without boundary with a product metric. Let $Q = \bigcup_j U_j$ be a finite cover of $Q$ by coordinate neighborhoods. Let $(x^1, x^2, \ldots, x^n) = (x^1, \bar{x})$ be coordinates on $(0, \infty) \times U_j$.

Let $h = \frac{1}{2}(x^1)^2$ for $c > 0$.

We define $d_h = e^h de^{-h}$, $d_h^* = e^{-h} de^h$, and

$$\Delta_h = d_h d_h^* + d_h^* d_h = \Delta + |dh|^2 + A_h,$$

all acting in $L^2(\Omega)$ (with measure $dx = \sqrt{\det g_{ij}} dx^i dx^j$ as usual). Then $\Delta_h$ on $L^2(\Omega)$ and $\Delta_\mu$ on $L^2(\mu \Omega)$ are unitarily-equivalent ([4]).

Denote $C_h^\infty = \bigcap_{k \geq 1} D(\Delta_h)^k$. Clearly, $\omega \in C_h^\infty$ iff $e^{-h} \omega \in C_\mu^\infty$. Smooth forms with compact support in the vertical direction are trivially in $C_h^\infty$. We will show that forms in $C_h^\infty$ have Schwartzian decay on a cylindrical end $E$ as $x^1 \to \infty$.

We prove the following result about forms in $C_h^\infty$:

**Theorem 16.** If $\omega \in C_h^\infty$ is supported in $E$, then for any $k \in \mathbb{N}$ and any multiindex $\alpha$,

$$|| (x^1)^k \partial^\alpha \omega ||_{L^2(\Omega)} < \infty. \quad (4)$$

Before proving (4), we note that it implies a pointwise result as well. That is, Sobolev inequalities (or even explicit Schwartz space calculations as in [22], Appendix to V.3) imply:

**Corollary 17.** If $\omega \in C_h^\infty$ is supported in $E$, then for any $k \in \mathbb{N}$, any multiindex $\alpha$, and any $\bar{x}_0 \in Q$,

$$|(x^1)^k \partial^\alpha \omega|(x^1, \bar{x}_0) \to 0 \quad \text{as } x^1 \to \infty. \quad (5)$$
Remark 18. Properties (4) and (5) do not depend on the choice of local coordinates $x$ on $Q$, since all transition functions and their derivatives are bounded on compact $Q$. We may always choose to work inside one of the coordinate neighborhoods.

Moreover, since inequality (4) is trivially true for smooth forms with compact support, we can assume (without loss of generality) that $\omega \equiv 0$ for $x^1 \leq 1$.

The proof of Theorem 16 will proceed in several steps, starting with following lemma.

Lemma 19. Weitzenbock’s formula for $\Delta_h$. If $\omega$ is a smooth form on $E$ then
\[ \Delta_h \omega = \left( \sum_{i=1}^n \nabla_i^* \nabla_i + R(\bar{x}) + c^2 (x^1)^2 \pm c \right) \omega, \] (6)
where $R(\bar{x}) : \wedge^p T^*_\bar{x} E \to \wedge^p T^*_\bar{x} E$ is a pointwise curvature endomorphism, and $\nabla_i = \nabla_{\partial_i / \partial x^i}$.

Proof. See [10], Chapter 12.4.

Lemma 20. Gårding’s inequality. Let $\omega$ be a smooth form, suppose $\text{supp} \, \omega \subset (1, \infty) \times Q$ and suppose $\omega, \Delta_h \omega \in L^2 \Omega((1, \infty) \times Q)$. Then
\[ \sum_{i=1}^n \| \nabla_i \omega \|^2 + \| x^1 \omega \|^2 \leq C \left( \langle \Delta_h \omega, \omega \rangle + \| \omega \|^2 \right). \] (7)

Proof. Let $J(\tau) : (0, \infty) \to \mathbb{R}_+$ be a smooth non-increasing function such that $J(\tau) \equiv 1$ for $\tau \leq 2$ and $J(\tau) \equiv 0$ for $\tau \geq 3$. We define a family of cut-off functions
\[ J_t : (0, \infty) \to \mathbb{R}_+ \]
by
\[ J_t(x^1, \bar{x}) = J \left( \frac{x^1}{t} \right). \]

We multiply both sides of (6) by $J_t^2 \omega$ and integrate by parts:
\[ \langle \Delta_h \omega, J_t^2 \omega \rangle = \sum_{i=1}^n \langle \nabla_i \omega, \nabla_i (J_t^2 \omega) \rangle + \langle R(\bar{x}) \omega, J_t^2 \omega \rangle + c^2 \| x^1 J_t \omega \|^2 \pm c \| J_t \omega \|^2. \] (8)
Since
\[ \langle \nabla_i \omega, \nabla_i (J_i^2 \omega) \rangle = \| J_i \nabla_i \omega \|^2 + 2 \langle J_i \frac{\partial J_i}{\partial x^j} \nabla_i \omega, \omega \rangle, \]
we have:
\[
\sum_{i=1}^n \| J_i \nabla_i \omega \|^2 + c^2 \| x^1 J_i \omega \|^2 \\
= \langle \Delta_h \omega, J_i^2 \omega \rangle - 2 \langle J_i \frac{\partial J_i}{\partial x^1} \nabla_1 \omega, \omega \rangle - \langle R(\bar{x}) \omega, J_i^2 \omega \rangle \pm c\| J_i \omega \|^2. \tag{9}
\]
Now we estimate:
\[
2\| \langle J_i \frac{\partial J_i}{\partial x^1} \nabla_1 \omega, \omega \rangle \| \leq \sup_{x^1} \left| \frac{\partial J_i}{\partial x^1} \right|^2 \| J_i \nabla_1 \omega \|^2 + \| \omega \|^2 \\
\leq \frac{C_1}{\ell^2} \| J_i \nabla_1 \omega \|^2 + \| \omega \|^2
\]
and
\[
\left| \langle R(\bar{x}) \omega, J_i^2 \omega \rangle \right| \leq \sup_{x} |R(\bar{x})|\| J_i \omega \|^2 \leq C_2 \| \omega \|^2.
\]
For large \( t \), the two inequalities above, together with (9), give:
\[
\sum_{i=1}^n \| J_i \nabla_i \omega \|^2 + c^2 \| x^1 J_i \omega \|^2 \leq C_3 \left( \langle \Delta_h \omega, J_i^2 \omega \rangle + \| J_i \omega \|^2 \right) \\
\leq C_3 \left( \langle \Delta_h \omega, \omega \rangle + \| \omega \|^2 + \langle \Delta_h \omega, (J_i^2 - 1) \omega \rangle \right). \tag{10}
\]
Gärding’s inequality follows if we take \( t \to \infty \) in the inequality above.

**Lemma 21.** Let \( \omega \) be a smooth form. Suppose \( \text{supp} \, \omega \subset (1, \infty) \times Q \) and suppose \( \omega, \Delta_h \omega \in L^2(1, \infty) \times Q \). Then
\[
\sum_{j=1}^n \| J_j \partial_j \omega \|^2 + \| J_j x^j \omega \|^2 \leq C \left( \| \langle J_j^2 \Delta_h \omega, \omega \rangle \| + \| J_j \omega \|^2 \right) \tag{11}
\]
and
\[
\sum_{j=1}^n \| \partial_j \omega \|^2 + \| x^j \omega \|^2 \leq C \left( \langle \Delta_h \omega, \omega \rangle + \| \omega \|^2 \right). \tag{12}
\]

**Proof.** Recall that for \( 2 \leq i \leq n \)
\[
\nabla_i (\omega_i dx^I) = (\nabla_i \omega_i) dx^I + \omega_i \nabla_i (dx^I) = (\partial_i \omega_I) dx^I + \omega_I \Gamma_i (\bar{x}) (dx^I), \tag{13}
\]
where $\Gamma_i(\bar{x}) : \wedge T^*_y \to \wedge T^*_x$ is an endomorphism. The matrix of $\Gamma_i(\bar{x})$ can be expressed in terms of the Christoffel symbols $\Gamma^k_{ij}(\bar{x})$ of a metric on the compact manifold $Q$.

Therefore
\[
\|J_i\partial_j\omega\|^2 \leq C_2 \left( \|J_i \nabla_j \omega\|^2 + \|J_i \omega\|^2 \right).
\] (15)

Inequality (11) now follow from (10) and (15). Inequality (12) follows from (11) as $t \to \infty$.

**Proof of Theorem 16. Induction argument.** For $\omega \in C^\infty_h$, supp $\omega \subset (1, \infty) \times Q$, inequalities (4) follow from
\[
\|\omega\|^2 := \sum_{k,\alpha} \| (x^1)^k \partial^\alpha \omega \|^2 \leq C(l) \left( \langle \Delta^l_h \omega, \omega \rangle + \|\omega\|^2 \right)
\] (16)

which must hold for all integers $0 \leq l < \infty$.

**Remark.** Operator $\Delta^l_h$ is a PDO of order $2l$ with smooth coefficients which grow at most as $(x_1)^{2l}$ as $x_1 \to \infty$. Therefore, for all $\omega \in C^\infty_h$ with supp $\omega \subset (1, \infty) \times Q$, which satisfy $\|\omega\|_l < \infty$, we have
\[
\langle \Delta^l_h \omega, \omega \rangle \leq C_l \|\omega\|^2.
\] (17)

Moreover, for all $l_0 \leq l$
\[
\langle \Delta^{l_0}_h \omega, \omega \rangle \leq C \left( \langle \Delta^l_h \omega, \omega \rangle + \|\omega\|^2 \right).
\]

We will prove (16) by induction in $l$. Lemma 21 proves (16) for $l = 1$.

Suppose (16) is true for all $l \leq l_0$ and all $\omega$ as above. Let $k + |\alpha| = l_0$. We substitute $(x^1)^k \partial^\alpha \omega$ instead of $\omega$ into (11) to get
\[
\sum_{j=1}^n \|J_i \partial_j (x^1)^k \partial^\alpha \omega\|^2 + \|J_i (x^1)^{k+1} \partial^\alpha \omega\|^2 \\
\leq C \left( \|J_i^2 \Delta_h (x^1)^k \partial^\alpha \omega, (x^1)^k \partial^\alpha \omega \| + \|J_i (x^1)^{k+1} \partial^\alpha \omega\|^2 \right).
\] (18)

Define $A = \langle J_i^2 \Delta_h (x^1)^k \partial^\alpha \omega, (x^1)^k \partial^\alpha \omega \rangle$. We write
\[
\Delta_h (x^1)^k \partial^\alpha \Delta_h \omega + \left[ \Delta_h, (x^1)^k \partial^\alpha \right] \omega,
\] (19)
where \([\cdot, \cdot]\) denotes the commutator of two differential operators.

From (6),
\[
\left[ \Delta_h, (x^1)^k \partial^\alpha \right] = \left[ \sum_{i=1}^n \nabla_i \nabla_i + R(x), (x^1)^k \partial^\alpha \right] + \epsilon^2 \left[ (x^1)^2, (x^1)^k \partial^\alpha \right] = P_1 + P_2. \tag{20}
\]

**Proposition.**

a) \(P_1\) is a PDO of order at most \(|\alpha| + 1\) with coefficients which grow at most as \((x^1)^k\).

b) \(P_2\) is a PDO of order at most \(|\alpha| - 1\) with coefficients which grow at most as \((x^1)^{k+1}\).

With the help of (19) and (20) we rewrite the quantity \(A\) as
\[
A = \langle J_t^2 (x^1)^k \partial^\alpha \Delta_h \omega, (x^1)^k \partial^\alpha \omega \rangle + \langle J_t^2 P_1 \omega, (x^1)^k \partial^\alpha \omega \rangle + \langle J_t^2 P_2 \omega, (x^1)^k \partial^\alpha \omega \rangle, \tag{21}
\]
and we estimate:
\[
|A| \leq \| J_t (x^1)^k \partial^\alpha \Delta_h \omega \| \cdot \| J_t (x^1)^k \partial^\alpha \omega \|
+ \epsilon \| J_t P_1 \omega \|^2 + \frac{1}{\epsilon} \| J_t (x^1)^k \partial^\alpha \omega \|^2 + \epsilon \| J_t P_2 \omega \|^2 + \frac{1}{\epsilon} \| J_t (x^1)^k \partial^\alpha \omega \|^2, \tag{22}
\]
where \(\epsilon > 0\) is to be chosen later. Now we estimate the right-hand side of (22).

For \(\Delta_h \omega \in C_0^\infty\), by induction and by the remark above we have
\[
\| J_t (x^1)^k \partial^\alpha \Delta_h \omega \| \cdot \| J_t (x^1)^k \partial^\alpha \omega \| \leq C_2 \left( \langle (\Delta_h)^{l_0+1} \omega, \omega \rangle + \| \omega \|^2 \right).
\]
Moreover, from the proposition,
\[
\| J_t P_1 \omega \|^2 + \| J_t P_2 \omega \|^2 \leq C_3 \sum_{m+|\beta| \leq l_0+1} \| J_t (x^1)^m \partial^\beta \omega \|^2.
\]
Therefore,
\[
|A| \leq C_4 \left( \langle (\Delta_h)^{l_0+1} \omega, \omega \rangle + \| \omega \|^2 + \epsilon \cdot \sum_{m+|\beta| \leq l_0+1} \| J_t (x^1)^m \partial^\beta \omega \|^2 + \| \omega \|_{l_0}^2 \right).
\]
The inequality (18) can be rewritten as
\[
\sum_{j=1}^n \| J_t (x^1)^k \partial_j \partial^\alpha \omega \|^2 + \| J_t (x^1)^{k+1} \partial^\alpha \omega \|^2
\leq C_4 \left( \langle (\Delta_h)^{l_0+1} \omega, \omega \rangle + \| \omega \|^2 + \| \omega \|_{l_0}^2 \right) + C_5 \epsilon \cdot \sum_{m+|\beta| \leq l_0+1} \| J_t (x^1)^m \partial^\beta \omega \|^2. \tag{23}
\]
We can sum up inequalities (23) over all \( k + \vert \alpha \vert = l_0 \) and use the equality
\[
\sum_{k+\alpha \leq l_0} \|J_t(x^1)^k \partial_j \partial^\alpha \omega\|^2 + \|J_t(x^1)^{k+1} \partial^\alpha \omega\|^2 = \sum_{m+\vert \beta \vert \leq l_0+1} \|J_t(x^1)^m \partial^\beta \omega\|^2
\]
to get (24) from (23):
\[
\sum_{m+\vert \beta \vert \leq l_0+1} \|J_t(x^1)^m \partial^\beta \omega\|^2 \leq C_6 (l_0 + 1) \left( (\Delta_t)^{l_0+1} \omega, \omega \right) + \|\omega\|^2 + \|\omega\|_{l_0}^2 \right) \\
+ \epsilon C_6 (l_0 + 1) \left( \sum_{m+\vert \beta \vert \leq l_0+1} \|J_t(x^1)^m \partial^\beta \omega\|^2 \right). (24)
\]
Finally, we choose \( \epsilon \) small enough to move the right-hand \( \sum_{m+\vert \beta \vert \leq l_0+1} \) over to the left. Then (16) follows from (24) for \( l = l_0 + 1 \) if we take \( t \to \infty \).

6. Applications

**Proposition 22.** Let \( M, N \) be orientable topologically tame manifolds. Then
\[
H_c(M \times N) \cong H_c(M) \otimes H_c(N).
\]

**Proof.** Choose metrics \( g_M, g_N \) so that \( M, N \) have cylindrical ends as in Theorem 2, and define \( d\mu_M, d\mu_N \) as described there, choosing \( c > 0 \). Apply Theorem 15 to \( M, N \), and note that the same methods used to prove Theorem 15 also show that \( H_c(M \times N) \cong H_{\mu_M \times \mu_N} \), where \( M \times N \) has been given the product metric and the product measure.

Now, \( H_{\mu_M \times \mu_N} \) has the expected combinatorics, coming purely from analysis on Hilbert tensor products—more precisely Fredholm complexes. See [6]. Specifically,
\[
H_{\mu_M \times \mu_N} \cong H_{\mu_M} \otimes H_{\mu_N},
\]
which proves our theorem.

The above theorem applies only to what [5] call manifolds with “finite good cover”, but it is nonetheless a Hodge-theoretic proof of a Künneth formula for certain noncompact manifolds. See [16], chapter 0, for proof of the Künneth formula on compact manifolds, using the same Hodge-theoretic technique of proof.
The Künneth formula in fact holds for any pair of manifolds \( M, F \) for which the cohomology of \( F \) is finite-dimensional.

Additionally, the results of our paper can be easily extended to the case of de Rham cohomology with values in a flat Hermitian bundle \( V \), provided that we put some conditions on \( V \) at infinity.

Suppose \( M \) is topologically tame and \( V \to M \) is a flat Hermitian vector bundle over \( M \) of rank \( d \). If \( \nabla : \Omega^\bullet(M, V) \to \Omega^\bullet(M, V) \) is a corresponding flat connection then \( \nabla^2 = 0 \) and we can define compactly supported de Rham cohomology with values in \( V \) from the complex \( (\Omega^\bullet(M, V), \nabla) \).

Our condition on \( V \) is that \( V \) be constant along each ray in the cylindrical end. Namely, let \( E = (0, \infty) \times Q \) be any cylindrical end of \( M \). We denote by \( p \) the natural projection \( p : (0, \infty) \times Q \to Q \). We require that the restriction \( V_{|E} \) of \( V \) to \( E \) is a pullback under \( p^* \) of the restriction \( V_{|Q} \) of \( V \) to \( Q \).

In addition we require the Hermitian metric \( h \) on \( V \) to have a restriction to \( V_{|E} \) that is flat along each ray \( (0, \infty) \times \{ \bar{x} \} \), \( \bar{x} \in Q \).

Under the conditions above, all proofs in Section 5 go through with the understanding that we now deal with the Witten Laplacian

\[
\Delta_h = \nabla_h \nabla_h^* + \nabla_h^* \nabla_h = \Delta + |dh|^2 + A_h,
\]

where \( \nabla_h = e^h \nabla e^{-h} \). Where applicable, \( | \bullet | \) denotes a Hermitian norm.

Finally, we observe that a differential form with values in a flat Hermitian bundle \( V \) carries some information about the fundamental group of \( M \) via a holonomy representation of this group. This additional information has proven to be useful in index theory ([17]), in studies of analytic torsion ([21]), and in generalizations of the Lefschetz formula to the case of flows ([13]).

References


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