GEOMETRY OF WARPED PRODUCTS AS RIEMANNIAN SUBMANIFOLDS AND RELATED PROBLEMS

BY

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Abstract. The warped product $N_1 \times_f N_2$ of two Riemannian manifolds $(N_1, g_1)$ and $(N_2, g_2)$ with warping function $f$ is the product manifold $N_1 \times N_2$ equipped with the warped product metric $g_1 + f^2 g_2$, where $f$ is a positive function on $N_1$. It is well-known that the notion of warped products plays some important roles in differential geometry as well as in physics. In this article we survey recent results on warped products isometrically immersed in real or complex space forms as Riemannian, Lagrangian, or CR-submanifolds. Moreover, we also present some recent related results concerning convolution of Riemannian manifolds, constant-ratio submanifolds, $T$-constant submanifolds, $N$-constant submanifolds and rectifying curves.

1. Introduction

Let $B$ and $F$ be two Riemannian manifolds of positive dimensions equipped with Riemannian metrics $g_B$ and $g_F$, respectively, and let $f$ be a positive function on $B$. Consider the product manifold $B \times F$ with its projection $\pi : B \times F \to B$ and $\eta : B \times F \to F$. The warped product $M = B \times_f F$ is the manifold $B \times F$ equipped with the Riemannian structure such that

$$||X||^2 = ||\pi_*(X)||^2 + f^2(\pi(x))||\eta_*(X)||^2,$$

(1.1)
for any tangent vector $X \in T_xM$. Thus, we have $g = g_B + f^2 g_F$. The function $f$ is called the *warping function* of the warped product (cf. [4]). It is well-known that the notion of warped products plays some important roles in differential geometry as well as in physics ([2, 49]). For instance, the best relativistic model of the Schwarzschild space-time that describes the out space around a massive star or a black hole is given as a warped product (cf. [49], pp.364-367).

One of the most fundamental problems in the theory of submanifolds is the immersibility (or non-immersibility) of a Riemannian manifold in a Euclidean space (or, more generally, in a space form). According to a well-known theorem of J. F. Nash, every Riemannian manifold can be isometrically immersed in some Euclidean spaces with sufficiently high codimension. Nash’s theorem implies in particular that *every warped product $N_1 \times_f N_2$ can be immersed as a Riemannian submanifold in some Euclidean space*. Moreover, many important submanifolds in real and complex space forms are expressed as warped products submanifolds, see, for instance, ([1, 4, 5, 6, 12, 14, 36, 37, 38, 39, 49, 50]).

In view of these simple facts, we propose the following very simple problem:

**Problem 1.**

$$\forall \ N_1 \times_f N_2 \xrightarrow{\text{isometric immersion}} \mathbb{E}^m \text{ or } R^m(c) \implies ??? \quad (1.2)$$

where $R^m(c)$ denotes a Riemannian $m$-manifold of constant sectional curvature $c$.

In particular, we ask the following.

**Problem 1.1.** Given a warped product $N_1 \times_f N_2$, what are the necessary conditions for the warped product to admit a minimal isometric immersion in a Euclidean $m$-space $\mathbb{E}^m$ (or in $R^m(c)$)?

**Problem 1.2.** Let $N_1 \times_f N_2$ be an arbitrary warped product immersed in a Euclidean space $\mathbb{E}^m$ as a Riemannian submanifold. What are the relationships between the warped product structure of $N_1 \times_f N_2$ and extrinsic structures of $N_1 \times_f N_2$ in $\mathbb{E}^m$ (or in $R^m(c)$) as a Riemannian submanifold?

Notice that the main warped structure of a warped product is warping function.
In this article, we present recent results and related results concerning these basic problems. More precisely, we survey recent results concerning warped products isometrically immersed in real or complex space forms as Riemannian, Lagrangian, or CR-submanifolds. Moreover, we also present some related results on convolution of Riemannian manifolds, constant-ratio submanifolds, $T$-constant submanifolds, $N$-constant submanifolds, as well as rectifying curves.

2. Basic Formulas

Let $N$ be an $n$-dimensional submanifold of a Riemannian $m$-manifold $R^m(c)$ of constant sectional curvature $c$. We choose a local field of orthonormal frame $e_1, \ldots, e_n, e_{n+1}, \ldots, e_m$ in $R^m(c)$ such that, restricted to $N$, the vectors $e_1, \ldots, e_n$ are tangent to $N$ and $e_{n+1}, \ldots, e_m$ are normal to $N$.

Let $K(e_i \wedge e_j), 1 \leq i < j \leq n$, denote the sectional curvature of the plane section spanned by $e_i, e_j$. Then the scalar curvature of $N$ is given by

$$\tau = \sum_{i<j} K(e_i \wedge e_j). \quad (2.1)$$

For a submanifold $N$ in $R^m(c)$ we denote by $\nabla$ and $\tilde{\nabla}$ the Levi-Civita connections of $N$ and $R^m(c)$, respectively. The Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_XY = \nabla_XY + \sigma(X,Y), \quad (2.2)$$

$$\tilde{\nabla}_X\xi = -A\xi + D_X\xi, \quad (2.3)$$

for vector fields $X, Y$ tangent to $N$ and vector field $\xi$ normal to $N$, where $\sigma$ denotes the second fundamental form, $D$ the normal connection, and $A$ the shape operator of the submanifold. Let $\{\sigma_{ij}^r\}, i, j = 1, \ldots, n; r = n+1, \ldots, m$, denote the coefficients of the second fundamental form with respect to $e_1, \ldots, e_n, e_{n+1}, \ldots, e_m$.

The mean curvature vector $\overline{H}$ is defined by

$$\overline{H} = \frac{1}{n}\text{trace } \sigma = \frac{1}{n} \sum_{i=1}^{n} \sigma(e_i, e_i), \quad (2.4)$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame of the tangent bundle $TN$ of $N$. The squared mean curvature is given by $H^2 = \langle \overline{H}, \overline{H} \rangle$, where $\langle , \rangle$ denotes
the inner product. A submanifold $N$ is called minimal in $R^m(c)$ if the mean curvature vector of $N$ in $R^m(c)$ vanishes identically.

Denote by $R$ the Riemann curvature tensors of $N$. Then the equation of Gauss is given by (see, for instance, [7])

$$R(X, Y; Z, W) = c\{\langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle \}$$

$$+ \langle \sigma(X, W), \sigma(Y, Z) \rangle - \langle \sigma(X, Z), \sigma(Y, W) \rangle,$$  \hspace{1cm} (2.5)

for vectors $X, Y, Z, W$ tangent to $N$.

Let $M$ be a Riemannian $p$-manifold and $e_1, \ldots, e_p$ be an orthonormal frame fields on $M$. For differentiable function $\varphi$ on $M$, the Laplacian $\Delta \varphi$ of $\varphi$ is defined by

$$\Delta \varphi = \sum_{j=1}^{p} \{\nabla_{e_j} e_j \varphi - \varphi e_j e_j \varphi\}.$$  \hspace{1cm} (2.6)

3. Riemannian Products, Moore’s Lemma and Nölker’s Theorem

Suppose that $M_1, \ldots, M_k$ are Riemannian manifolds and that

$$f : M_1 \times \cdots \times M_k \to E^N$$

is an isometric immersion of the Riemannian product $M_1 \times \cdots \times M_k$ into Euclidean $N$-space. J. D. Moore [42] proved that if the second fundamental form $h$ of $f$ has the property that $h(X, Y) = 0$ for $X$ tangent to $M_i$ and $Y$ tangent to $M_j$, $i \neq j$, then $f$ is a product immersion, that is, there exist isometric immersions $f_i : M_i \to E^{m_i}$, $1 \leq i \leq k$, such that

$$f(x_1, \ldots, x_k) = (f(x_1), \ldots, f(x_k)),$$  \hspace{1cm} (3.1)

when $x_i \in M_i$ for $1 \leq i \leq k$.

Let $M_0, \cdots, M_k$ be Riemannian manifolds, $M = M_0 \times \cdots \times M_k$ their product, and $\pi_i : M \to M_i$ the canonical projection. If $\rho_1, \cdots, \rho_k : M_0 \to \mathbb{R}_+$ are positive-valued functions, then

$$\langle X, Y \rangle := \langle \pi_0 X, \pi_0 Y \rangle + \sum_{i=1}^{k} (\rho_i \circ \pi_0)^2 \langle \pi_i X, \pi_i Y \rangle$$  \hspace{1cm} (3.2)
defines a Riemannian metric on $M$, called a warped product metric. $M$ endowed with this metric is denoted by $M_0 \times_{\rho_1} M_1 \times \cdots \times_{\rho_k} M_k$.

A warped product immersion is defined as follows: Let $M_0 \times_{\rho_1} M_1 \times \cdots \times_{\rho_k} M_k$ be a warped product and let $f_i : N_i \to M_i$, $i = 0, \cdots, k$, be isometric immersions, and define $\sigma_i := \rho_i \circ f_0 : N_0 \to \mathbb{R}_+$ for $i = 1, \cdots, k$. Then the map

$$f : N_0 \times_{\sigma_1} N_1 \times \cdots \times_{\sigma_k} N_k \to M_0 \times_{\rho_1} M_1 \times \cdots \times_{\rho_k} M_k$$

(3.3)

given by $f(x_0, \cdots, x_k) := (f_0(x_0), f_1(x_1), \cdots, f_k(x_k))$ is an isometric immersion, which is called a warped product immersion.

S. Nölker [47] extended Moore’s result to the following.

Nölker Theorem. Let $f : N_0 \times_{\sigma_1} N_1 \times \cdots \times_{\sigma_k} N_k \to \mathbb{R}^N(c)$ be an isometric immersion into a Riemannian manifold of constant curvature $c$. If $h$ is the second fundamental form of $f$ and $h(X_i, X_j) = 0$, for all vector fields $X_i$ and $X_j$, tangent to $N_i$ and $N_j$ respectively, with $i \neq j$, then, locally, $f$ is a warped product immersion.

Warped product immersions was applied in [38] by Dillen and Nölker to classify semi-parallel immersions with flat normal connection. For a recent extensions of Nölker theorem, see [42, 51].

4. Warped Products in Real Space Forms

In this section we present some recent general results concerning warped products isometrically immersed in real space forms. These results provide some solutions to Problem 1.

Theorem 4.1.([28]) For any isometric immersion $\phi : N_1 \times f N_2 \to \mathbb{R}^m(c)$ of a warped product $N_1 \times f N_2$ into a Riemannian manifold of constant curvature $c$, we have

$$\frac{\Delta f}{f} \leq \frac{n^2 c}{4n_2} + n_1 c,$$

(4.1)

where $n_i = \dim N_i$, $n = n_1 + n_2$, $H^2$ is the squared mean curvature of $\phi$, and $\Delta f$ is the Laplacian $f$ on $N_1$. 
The equality sign of (4.1) holds identically if and only if there exists a warped product representation $M_1 \times _\rho M_2$ of $R^m(c)$ such that $\phi : N_1 \times f \ N_2 \to M_1 \times _\rho M_2$ is a minimal warped product immersion.

**Theorem 4.2.** ([27]) For any isometric immersion $\phi : N_1 \times f \ N_2 \to R^m(c)$, the scalar curvature $\tau$ of the warped product $N_1 \times f \ N_2$ satisfies

$$\tau \leq \frac{\Delta f}{n_1 f} + \frac{n^2(n - 2)}{2(n - 1)} H^2 + \frac{1}{2}(n + 1)(n - 2)c. \quad (4.2)$$

If $n = 2$, the equality case of (4.2) holds automatically.

If $n \geq 3$, the equality sign of (4.2) holds identically if and only if either

(a) $N_1 \times f \ N_2$ is of constant curvature $c$, the warping function $f$ is an eigenfunction with eigenvalue $c$, i.e., $\Delta f = cf$, and $N_1 \times f \ N_2$ is immersed as a totally geodesic submanifold in $R^m(c)$, or

(b) locally, $N_1 \times f \ N_2$ is immersed as a rotational hypersurface in a totally geodesic submanifold $R^{n+1}(c)$ of $R^m(c)$ with a geodesic of $R^{n+1}(c)$ as its profile curve.

**Remark 4.1.** Every Riemannian manifold of constant curvature $c$ can be locally expressed as a warped product whose warping function satisfies $\Delta f = cf$. For examples, $S^n(1)$ is locally isometric to $(0, \infty) \times _{\cos x} S^{n-1}(1)$; $E^n$ is locally isometric to $(0, \infty) \times _x S^{n-1}(1)$; and $H^n(-1)$ is locally isometric to $\mathbb{R} \times _{e^x} E^{n-1}$.

**Remark 4.2.** There are other warped product decompositions of $R^n(c)$ whose warping function satisfies $\Delta f = cf$. For example, let $\{x_1, \ldots, x_{n_1}\}$ be a Euclidean coordinate system of $E^{n_1}$ and let $\rho = \sum_{j=1}^{n_1} a_j x_j + b$, where $a_1, \ldots, a_{n_1}, b$ are real numbers satisfying $\sum_{j=1}^{n_1} a_j^2 = 1$. Then the warped product $E^{n_1} \times _\rho S^{n_2}(1)$ is a flat space whose warping function is a harmonic function.

In fact, those are the only warped product decompositions of flat spaces whose warping functions are harmonic functions.

**Remark 4.3.** Ejiri constructed in [39] many examples of minimal warped product immersions into Riemannian manifolds of constant curvature. Moreover, he also proved the following.
Theorem 4.3. ([40]) There exist countably many immersions of $S^1 \times S^{n-1}$ into $S^{n+1}$ such that the induced metric on $S^1 \times S^{n-1}$ is a warped product of constant scalar curvature $n(n - 1)$.

Now, we provide some immediate applications.

Corollary 4.1. ([28]) Let $N_1$ and $N_2$ be two Riemannian manifold and $f$ be a nonzero harmonic function on $N_1$. Then every minimal isometric immersion of $N_1 \times f N_2$ into any Euclidean space is a warped product immersion.

Remark 4.4. There exist many such minimal immersions. For example, if $N_2$ is a minimal submanifold of $S^{m-1}(1) \subset \mathbb{E}^m$, then the minimal cone $C(N_2)$ over $N_2$ with vertex at the origin is a warped product $\mathbb{R}_+ \times s N_2$ whose warping function $f = s$ is a harmonic function. Here $s$ is the coordinate function of the positive real line $\mathbb{R}_+$.

Corollary 4.2. ([28]) Let $f \neq 0$ be a harmonic function on $N_1$. Then for any Riemannian manifold $N_2$ the warped product $N_1 \times f N_2$ does not admits any minimal isometric immersion into any hyperbolic space.

Corollary 4.3. ([28]) Let $f$ be a function on $N_1$ with $\Delta f = \lambda f$ with $\lambda > 0$. Then for any Riemannian manifold $N_2$ the warped product $N_1 \times f N_2$ admits no minimal isometric immersion into any Euclidean space or hyperbolic space.

Remark 4.5. In views of Corollaries 4.2 and 4.3, we point out that there exist minimal immersions from $N_1 \times f N_2$ into hyperbolic space such that the warping function $f$ is an eigenfunction with negative eigenvalue. For example, $\mathbb{R} \times e^x \mathbb{E}^{n-1}$ admits an isometric minimal immersion into the hyperbolic space $H^{n+1}(-1)$ of constant curvature $-1$.

Corollary 4.4. ([28]) If $N_1$ is a compact, then $N_1 \times f N_2$ does not admit a minimal isometric immersion into any Euclidean space or hyperbolic space.

Remark 4.6. For Corollary 4.4, we point out that there exist many minimal immersions of $N_1 \times f N_2$ into $\mathbb{E}^m$ with compact $N_2$. For examples, a hypercater- noid in $\mathbb{E}^{n+1}$ is a minimal hypersurfaces which is isometric to a warped product $\mathbb{R} \times f S^{n-1}$. Also, for any compact minimal submanifold $N_2$ of $S^{m-1} \subset \mathbb{E}^m$, the minimal cone $C(N_2)$ is a warped product $\mathbb{R}_+ \times s N_2$ which is also a such example.
Remark 4.7. Contrast to Euclidean and hyperbolic spaces, $S^m$ admits minimal warped product submanifolds $N_1 \times_f N_2$ with $N_1$ and $N_2$ being compact. The simplest such examples are minimal Clifford tori defined by

$$M_{k,n-k} = S^k \left( \sqrt{\frac{k}{n}} \right) \times S^{n-k} \left( \sqrt{\frac{n-k}{n}} \right) \subset S^{n+1}, \ 1 \leq k < n.$$ 

Remark 4.8. In [54] Suceava construct a family of warped products of hyperbolic planes which do not admit any isometric minimal immersion into Euclidean space, by applying the invariants introduced in [15].

5. Warped Product Real Hypersurfaces in Complex Space Forms

For Riemannian product real hypersurfaces in complex space forms, we have the following non-existence theorem.

Theorem 5.1. ([32]) There do not exist real hypersurfaces in complex projective and complex hyperbolic spaces which are Riemannian products of two or more Riemannian manifolds of positive dimension.

In other words, every real hypersurface in a nonflat complex space form is irreducible.

A contact manifold is an odd-dimensional manifold $M^{2n+1}$ with a 1-form $\eta$ such that $\eta \wedge (d\eta)^n \neq 0$. A curve $\gamma = \gamma(t)$ in a contact manifold is called a Legendre curve if $\eta(\beta'(t)) = 0$ along $\beta$. Let $S^{2n+1}(c)$ denote the hypersphere in $C^{n+1}$ with curvature $c$ centered at the origin. Then $S^{2n+1}(c)$ is a contact manifold endowed with a canonical contact structure which is the dual 1-form of the characteristic vector field $J \xi$, where $J$ is the complex structure and $\xi$ the unit normal vector on $S^{2n+1}(c)$.

Legendre curves are known to play an important role in the study of contact manifolds, e.g. a diffeomorphism of a contact manifold is a contact transformation if and only if it maps Legendre curves to Legendre curves.

Contrast to Theorem 5.1, there exist many warped product real hypersurfaces in complex space forms as given in the following three theorems.
Theorem 5.2. ([24]) Let $a$ be a positive number and $\gamma(t) = (\Gamma_1(t), \Gamma_2(t))$ be a unit speed Legendre curve $\gamma : I \to S^3(a^2) \subset \mathbb{C}^2$ defined on an open interval $I$. Then
\[
x(z_1, \ldots, z_n, t) = (a\Gamma_1(t)z_1, a\Gamma_2(t)z_1, z_2, \ldots, z_n), \quad z_1 \neq 0,
\] defines a real hypersurface which is the warped product $\mathbb{C}^n_{a} \times_{a[z_1]} I$ of a complex $n$-plane and $I$, where $\mathbb{C}^n_{a} = \{(z_1, \ldots, z_n) : z_1 \neq 0\}$.

Conversely, up to rigid motions of $\mathbb{C}^{n+1}$, every real hypersurface in $\mathbb{C}^{n+1}$ which is the warped product $N \times_f I$ of a complex hypersurface $N$ and an open interval $I$ is either obtained in the way described above or given by the product submanifold $\mathbb{C}^n \times C \subset \mathbb{C}^n \times C^1$ of $\mathbb{C}^n$ and a real curve $C$ in $C^1$.

Let $S^{2n+3}$ denote the unit hypersphere in $\mathbb{C}^{n+2}$ centered at the origin and put $U(1) = \{\lambda \in \mathbb{C} : \lambda\bar{\lambda} = 1\}$. Then there is a $U(1)$-action on $S^{2n+3}$ defined by $z \mapsto \lambda z$. At $z \in S^{2n+3}$ the vector $V = iz$ is tangent to the flow of the action. The quotient space $S^{2n+3}/\sim$, under the identification induced from the action, is a complex projective space $CP^{n+1}$ which endows with the canonical Fubini-Study metric of constant holomorphic sectional curvature 4. The almost complex structure $J$ on $CP^{n+1}$ is induced from the complex structure $J$ on $\mathbb{C}^{n+2}$ via the Hopf fibration: $\pi : S^{2n+3} \to CP^{n+1}$. It is well-known that the Hopf fibration $\pi$ is a Riemannian submersion such that $V = iz$ spans the vertical subspaces.

Let $\phi : M \to CP^{n+1}$ be an isometric immersion. Then $\hat{\phi} = \pi^{-1}(M)$ is a principal circle bundle over $M$ with totally geodesic fibers. The lift $\hat{\phi} : \hat{M} \to S^{2n+3}$ of $\phi$ is an isometric immersion so that the diagram:
\[
\begin{array}{ccc}
\hat{M} & \xrightarrow{\hat{\phi}} & S^{2n+3} \\
\pi \downarrow & & \downarrow \pi \\
M & \to & CP^{n+1}
\end{array}
\]
commutes.

Conversely, if $\psi : \hat{M} \to S^{2n+3}$ is an isometric immersion which is invariant under the $U(1)$-action, then there is a unique isometric immersion $\psi_\pi : \pi(\hat{M}) \to$
$CP^{n+1}$ such that the associated diagram commutes. We simply call the immersion $\psi_\pi : \pi(\hat{M}) \rightarrow CP^{n+1}$ the projection of $\psi : \hat{M} \rightarrow S^{2n+3}$.

For a given vector $X \in T_z(CP^{n+1})$ and a point $u \in S^{2n+2}$ with $\pi(u) = z$, we denote by $X_u^*$ the horizontal lift of $X$ at $u$ via $\pi$. There exists a canonical orthogonal decomposition:

$$T_uS^{2n+3} = (T_{\pi(u)}CP^{n+1})_u \oplus \text{Span}\{V_u\}. \quad (5.2)$$

Since $\pi$ is a Riemannian submersion, $X$ and $X_u^*$ have the same length.

We put

$$S^{2n+1}_s = \left\{(z_0, \ldots, z_n) : \sum_{k=0}^{n} z_k\bar{z}_k = 1, z_0 \neq 0\right\}, \quad CP^{n}_0 = \pi(S^{2n+1}_s). \quad (5.3)$$

**Theorem 5.3.** ([24]) Suppose that $a$ is a positive number and $\gamma(t) = (\Gamma_1(t), \Gamma_2(t))$ is a unit speed Legendre curve $\gamma : I \rightarrow S^3(a^2) \subset \mathbb{C}^2$ defined on an open interval $I$. Let $x : S^{2n+1}_s \times I \rightarrow \mathbb{C}^{n+2}$ be the map defined by

$$x(z_0, \ldots, z_n, t) = (a\Gamma_1(t)z_0, a\Gamma_2(t)z_0, z_1, \ldots, z_n), \quad \sum_{k=0}^{n} z_k\bar{z}_k = 1. \quad (5.4)$$

Then

(i) $x$ induces an isometric immersion $\psi : S^{2n+1}_s \times_{a|z_0|} I \rightarrow S^{2n+3}$.

(ii) The image $\psi(S^{2n+1}_s \times_{a|z_0|} I)$ in $S^{2n+3}$ is invariant under the action of $U(1)$.

(iii) the projection $\psi_\pi : \pi(S^{2n+1}_s \times_{a|z_0|} I) \rightarrow CP^{n+1}$ of $\psi$ via $\pi$ is a warped product hypersurface $CP^{n}_0 \times_{a|z_0|} I$ in $CP^{n+1}$.

Conversely, if a real hypersurface in $CP^{n+1}$ is a warped product $N \times_I I$ of a complex hypersurface $N$ of $CP^{n+1}$ and an open interval $I$, then, up to rigid motions, it is locally obtained in the way described above.

In the complex pseudo-Euclidean space $\mathbb{C}_1^{n+2}$ endowed with pseudo-Euclidean metric

$$g_0 = -dz_0d\bar{z}_0 + \sum_{j=1}^{n+1} dz_jd\bar{z}_j, \quad (5.5)$$

we define the anti-de Sitter space-time by

$$H^{2n+3}_1 = \{(z_0, z_1, \ldots, z_{n+1}) : \langle z, z \rangle = -1\}. \quad (5.6)$$
It is known that $H^{2n+3}_1$ has constant sectional curvature $-1$. There is a $U(1)$-action on $H^{2n+3}_1$ defined by $z \mapsto \lambda z$. At a point $z \in H^{2n+3}_1$, $iz$ is tangent to the flow of the action. The orbit is given by $z_t = e^{it}z$ with $\frac{dz_t}{dt} = iz_t$ which lies in the negative-definite plane spanned by $z$ and $iz$. The quotient space $H^{2n+3}_1/\sim$ is the complex hyperbolic space $CH^{n+1}$ which endows a canonical Kähler metric of constant holomorphic sectional curvature $-4$. The complex structure $J$ on $CH^{n+1}$ is induced from the canonical complex structure $J$ on $\mathbb{C}^{n+2}$ via the totally geodesic fibration:

Let $\phi : M \to CH^{n+1}$ be an isometric immersion. Then $\hat{M} = \pi^{-1}(M)$ is a principal circle bundle over $M$ with totally geodesic fibers. The lift $\hat{\phi} : \hat{M} \to H^{2n+3}_1$ of $\phi$ is an isometric immersion such that the diagram:

\[
\begin{array}{ccc}
\hat{M} & \xrightarrow{\hat{\phi}} & H^{2n+3}_1 \\
\downarrow{\pi} & & \downarrow{\pi} \\
M & \xrightarrow{\phi} & CH^{n+1}
\end{array}
\]

commutes.

Conversely, if $\psi : \hat{M} \to H^{2n+3}_1$ is an isometric immersion which is invariant under the $U(1)$-action, there is a unique isometric immersion $\psi_\pi : \pi(\hat{M}) \to CH^{n+1}$, called the projection of $\psi$ so that the associated diagram commutes. We put

$$H^{2n+1}_{1s} = \{(z_0, \ldots, z_n) \in H^{2n+1}_1 : z_n \neq 0\},$$

$$CH^n_\pi = \pi(H^{2n+1}_{1s}).$$

**Theorem 5.4.** ([24]) Suppose that $\alpha$ is a positive number and $\gamma(t) = (\Gamma_1(t), \Gamma_2(t))$ is a unit speed Legendre curve $\gamma : I \to S^3(\alpha^2) \subset \mathbb{C}^2$. Let $y : H^{2n+1}_{1s} \times I \to \mathbb{C}^{n+2}$ be the map defined by

$$y(z_0, \ldots, z_n, t) = (z_0, \ldots, z_{n-1}, \alpha \Gamma_1(t) z_n, \alpha \Gamma_2(t) z_n),$$

$$z_0 \bar{z}_0 - \sum_{k=1}^{n} z_k \bar{z}_k = 1.$$  

Then
(i) **y** induces an isometric immersion \( \psi : H_1^{2n+1} \times_a[z_n] I \rightarrow H_1^{2n+3} \).

(ii) The image \( \psi(H_1^{2n+1} \times_a[z_n] I) \) in \( H_1^{2n+3} \) is invariant under the \( U(1) \)-action.

(iii) The projection \( \psi_\pi : \pi(H_1^{2n+1} \times_a[z_n] I) \rightarrow CH^{n+1} \) of \( \psi \) via \( \pi \) is a warped product hypersurface \( CH_n^a \times_a[z_n] I \) in \( CH^{n+1} \).

Conversely, if a real hypersurface in \( CH^{n+1} \) is a warped product \( N \times f I \) of a complex hypersurface \( N \) and an open interval \( I \), then, up to rigid motions, it is locally obtained in the way described above.

### 6. Warped Products as Lagrangian Submanifolds

A submanifold \( N \) in a Kähler manifold \( \tilde{M} \) is called a totally real submanifold if the almost complex structure \( J \) of \( \tilde{M} \) carries each tangent space \( T_x N \) of \( N \) into its corresponding normal space \( T_x^\perp N \) [34]. The submanifold \( N \) is called a holomorphic submanifold (or Kähler submanifold) if \( J \) carries each \( T_x N \) into itself [48].

A totally real submanifold \( N \) in a Kähler manifold \( \tilde{M} \) is called a Lagrangian submanifold if \( \dim_\mathbb{R} N = \dim_\mathbb{C} \tilde{M} \). (For the most recent survey on differential geometry of Lagrangian submanifolds, see [17, 22].)

For Lagrangian immersions into complex Euclidean \( n \)-space \( \mathbb{C}^n \), a well-known result of M. Gromov [41] states that a compact \( n \)-manifold \( M \) admits a Lagrangian immersion (not necessary isometric) into \( \mathbb{C}^n \) if and only if the complexification \( TM \otimes \mathbb{C} \) of the tangent bundle of \( M \) is trivial. In particular, Gromov’s result implies that there exists no topological obstruction to Lagrangian immersions for compact 3-manifolds in \( \mathbb{C}^3 \), because the tangent bundle of a 3-manifold is always trivial.

From the Riemannian point of view, it is natural to ask the following basic question.

**Problem 6.1.** When a Riemannian \( n \)-manifold admits a Lagrangian isometric immersion into \( \mathbb{C}^n \)?

Not every warped product \( N_1 \times f N_2 \) can be isometrically immersed in a complex space form as a Lagrangian submanifold. However, for warped products of curves and the unit \((n - 1)\)-sphere \( S^{n-1} \), we have the following existence theorem.
**Theorem 6.1.** ([13, 18]) *Every simply-connected open portion of a warped product manifold* $I \times_f S^{n-1}$ *of an open interval* $I$ *and a unit* $(n - 1)$-*sphere admits an isometric Lagrangian immersion into* $\mathbb{C}^n$.

The Lagrangian immersions given in Theorem 6.1 with vanishing scalar curvature have been recently determined explicitly in [5].

The Lagrangian immersions given in Theorem 6.1 are expressed in terms of complex extensors in the sense of [13] which are defined as follows:

Let $G : M^{n-1} \to \mathbb{E}^m$ be an isometric immersion of a Riemannian $(n - 1)$-manifold into Euclidean $m$-space $\mathbb{E}^m$ and $F : I \to \mathbb{C}$ a unit speed curve in the complex plane. We extend the immersion $G : M^{n-1} \to \mathbb{E}^m$ to an immersion of $I \times M^{n-1}$ into complex Euclidean $m$-space $\mathbb{C}^m$ given by

$$
\phi = F \otimes G : I \times M^{n-1} \to \mathbb{C} \otimes \mathbb{E}^m = \mathbb{C}^m,
$$

where $F \otimes G$ is the tensor product immersion of $F$ and $G$ defined by

$$
(F \otimes G)(s, p) = F(s) \otimes G(p), \quad s \in I, \ p \in M^{n-1}.
$$

Such an extension $F \otimes G$ of the immersion $G$ is called a complex extensor of $G$ (or of submanifold $M^{n-1}$) via $F$.

Those Lagrangian immersions obtained in Theorem 6.1 show how to embed a time slice of the Schwarzschild space-time that models the out space around a massive star or a black hole as Lagrangian submanifolds (cf. [5, 49]).

Since rotation hypersurfaces and real space forms can be at least locally expressed as the warped products of curves and a unit sphere, Theorem 6.1 implies immediately the following

**Corollary 6.1.** *Every rotation hypersurface of* $\mathbb{E}^{n+1}$ *can be isometrically immersed as Lagrangian submanifold in* $\mathbb{C}^n$.

**Corollary 6.2.** *Every Riemannian n-manifold of constant sectional curvature* $c$ *can be locally isometrically immersed in* $\mathbb{C}^n$ *as a Lagrangian submanifold.*

**Corollary 6.3.** *Every rotation Every simply-connected surface equipped with a Riemannian metric* $g = E^2(x)(dx^2 + dy^2)$ *always admits a Lagrangian isometric immersion into* $\mathbb{C}^2$. 
Remark 6.1. Not every Riemannian \( n \)-manifold of constant sectional curvature can be globally isometrically immersed in \( \mathbb{C}^n \) as a Lagrangian submanifold. For instance, it is known from [15] that every compact Riemannian \( n \)-manifold with positive sectional curvature (or positive Ricci curvature) does not admit any Lagrangian isometric immersion into \( \mathbb{C}^n \).

Remark 6.2. In views of Theorem 6.1 it is interesting to ask the following.

Problem 6.2. Determine necessary and/or sufficient conditions for warped product manifolds to admit Lagrangian isometric immersions into complex Euclidean spaces or more generally into complex space forms.

7. Segre Embedding and Its Extensions

Let \( (z^i_0, \ldots, z^i_{\alpha_i}) \) \( (1 \leq i \leq s) \) denote the homogeneous coordinates of \( \mathbb{C}P^{\alpha_i} \). Define a map:

\[
S_{\alpha_1, \ldots, \alpha_s} : \mathbb{C}P^{\alpha_1} \times \cdots \times \mathbb{C}P^{\alpha_s} \to \mathbb{C}P^N, \quad N = \prod_{i=1}^s (\alpha_i + 1) - 1,
\]

which maps a point \( ((z^1_0, \ldots, z^1_{\alpha_1}), \ldots, (z^s_0, \ldots, z^s_{\alpha_s})) \) of the product Kähler manifold \( \mathbb{C}P^{\alpha_1} \times \cdots \times \mathbb{C}P^{\alpha_s} \) to the point \( (z^1_{i_1} \cdots z^s_{i_s})_{1 \leq i_1 \leq \alpha_1, \ldots, 1 \leq i_s \leq \alpha_s} \) in \( \mathbb{C}P^N \). The map \( S_{\alpha_1, \ldots, \alpha_s} \) is a Kähler embedding which is called the Segre embedding. The Segre embedding was constructed by C. Segre in 1891 [52].

The following results obtained in 1981 can be regarded as “converse” to Segre embedding constructed in 1891.

Theorem 7.1. ([19, 32]) Let \( M^{\alpha_1}_1, \ldots, M^{\alpha_s}_s \) be Kähler manifolds of dimensions \( \alpha_1, \ldots, \alpha_s \), respectively. Then every Kähler immersion

\[
f : M^{\alpha_1}_1 \times \cdots \times M^{\alpha_s}_s \to \mathbb{C}P^N, \quad N = \prod_{i=1}^s (\alpha_i + 1) - 1,
\]

of \( M^{\alpha_1}_1 \times \cdots \times M^{\alpha_s}_s \) into \( \mathbb{C}P^N \) is locally the Segre embedding, that is, \( M^{\alpha_1}_1, \ldots, M^{\alpha_s}_s \) are open portions of \( \mathbb{C}P^{\alpha_1}, \ldots, \mathbb{C}P^{\alpha_s} \), respectively, and moreover, the Kähler immersion \( f \) is congruent to the Segre embedding.

This theorem was proved in [44] under two additional assumptions; namely, \( s = 2 \) and the Kähler immersion \( f \) has parallel second fundamental form.
Let $\nabla^k \sigma$ denote the $k$-th covariant derivative of the second fundamental form for $k = 0, 1, 2, \cdots$. Denoted by $|\nabla^k \sigma|^2$ the squared norm of $\nabla^k \sigma$.

**Theorem 7.2.** ([19, 32]) Let $M_1^{\alpha_1} \times \cdots \times M_s^{\alpha_s}$ be a product Kähler submanifold of $CP^N$. Then we have

$$||\nabla^{k-2} \sigma||^2 \geq k! 2^k \sum_{i_1 < \cdots < i_k} \alpha_1 \cdots \alpha_k,$$  \hspace{1cm} (7.2)

for $k = 2, 3, \cdots$.

The equality sign of (7.2) holds for some $k$ if and only if $M_1^{\alpha_1}, \ldots, M_s^{\alpha_s}$ are open portions of $CP^{\alpha_1}, \ldots, CP^{\alpha_s}$, respectively, and the Kähler immersion is congruent to the Segre embedding.

In particular, if $k = 2$, Theorem 7.2 reduces to

**Theorem 7.3.** ([19, 32]) Let $M_1^{h} \times M_2^{p}$ be a product Kähler submanifold of $CP^N$. Then we have

$$||\sigma||^2 \geq 8hp.$$  \hspace{1cm} (7.3)

The equality sign of (7.3) holds if and only if $M_1^{h}$ and $M_2^{p}$ are open portions of $CP^{h}$ and $CP^{p}$, respectively, and the Kähler immersion is congruent to the Segre embedding $S_h,p$.

**8. CR-Products**

A submanifold $N$ in a Kähler manifold $\tilde{M}$ is called a $CR$-submanifold [3] if there exists on $N$ a holomorphic distribution $\mathcal{D}$ whose orthogonal complement $\mathcal{D}^\perp$ is a totally real distribution, i.e., $J\mathcal{D}^\perp \subset T_x^\perp N$.

A $CR$-submanifold of a Kähler manifold $\tilde{M}$ is called a $CR$-product [8] if it is a Riemannian product $N_K \times N_\perp$ of a Kähler submanifold $N_K$ and a totally real submanifold $N_\perp$.

For $CR$-products in complex space forms, the following are known.

**Theorem 8.1.** ([8]) We have

(i) A $CR$-submanifold in $C^n$ is a $CR$-product if and only if it is a direct sum of a Kähler submanifold and a totally real submanifold of linear complex subspaces.
(ii) There do not exist CR-products in complex hyperbolic spaces other than Kähler submanifolds and totally real submanifolds.

CR-products $N_K \times N_\perp$ in $CP^{h+p+hp}$ are obtained from the Segre embedding; namely, we have the following.

**Theorem 8.2.** ([8]) Let $N^h_T \times N^p_\perp$ be the CR-product in $CP^m$ with constant holomorphic sectional curvature 4. Then

$$m \geq h + p + hp.$$  \hspace{1cm} (8.1)

The equality sign of (8.1) holds if and only if
(a) $N^h_T$ is a totally geodesic Kähler submanifold,
(b) $N^p_\perp$ is a totally geodesic totally real submanifold, and
(c) the immersion is given by

$$N^h_T \times N^p_\perp \xrightarrow{\text{t.g.}} CP^h \times CP^p \xrightarrow{S_{hp}} Segre\ imbedding \ CP^{h+p+hp}. \hspace{1cm} (8.2)$$

**Theorem 8.3.** ([8]) Let $N^h_T \times N^p_\perp$ be the CR-product in $CP^m$ with constant holomorphic sectional curvature 4. Then the squared norm of the second fundamental form satisfies

$$||\sigma||^2 \geq 4hp.$$  \hspace{1cm} (8.3)

The equality sign of (8.3) holds if and only if
(a) $N^h_T$ is a totally geodesic Kähler submanifold,
(b) $N^p_\perp$ is a totally geodesic totally real submanifold, and
(c) the immersion is given by

$$N^h_T \times N^p_\perp \xrightarrow{\text{t.g.}} CP^h \times CP^p \xrightarrow{S_{hp}} Segre\ imbedding \ CP^{h+p+hp} \subset CP^m. \hspace{1cm} (8.4)$$

9. Warped Products as CR-Submanifolds

For warped product CR-submanifolds in complex space forms, we propose the following simple question:
Problem 9.1.

\[ \forall \begin{cases} N_K \times_f N_\perp \text{ CR-immersion} & \rightarrow \mathbb{C}^m \text{ or } \tilde{M}^m(4c) \implies \text{???} \\ N_\perp \times_f N_N \end{cases} \]

where \( \tilde{M}^m(4c) \) denotes a Kähler manifold of constant holomorphic sectional curvature \( 4c \).

First we mention the following.

**Theorem 9.1.** ([19]) If \( N_\perp \times_f N_K \) is a warped product CR-submanifold of a Kähler manifold \( \tilde{M} \) such that \( N_\perp \) is a totally real and \( N_K \) a Kähler submanifold of \( \tilde{M} \), then it is a CR-product.

Theorem 9.1 shows that there does not exist warped product CR-submanifolds of the form \( N_\perp \times_f N_K \) other than CR-products. So, we shall only consider warped product CR-submanifolds of the form: \( N_K \times_f N_\perp \), by reversing the two factors \( N_K \) and \( N_\perp \). We simply call such CR-submanifolds CR-warped products.

CR-warped products are characterized as follows.

**Theorem 9.2.** ([19]) A proper CR-submanifold \( M \) of a Kähler manifold \( \tilde{M} \) is locally a CR-warped product if and only if

\[ A_{1Z}X = ((JX)\mu)Z, \quad X \in \mathcal{D}, \quad Z \in \mathcal{D}^\perp, \]

for some function \( \mu \) on \( M \) satisfying \( W\mu = 0, W \in \mathcal{D}^\perp \).

We have the following solution to Problem 9.1.

**Theorem 9.3.** ([19]) Let \( N_K \times_f N_\perp \) be a CR-warped product in a Kähler manifold \( \tilde{M} \). Then

\[ ||\sigma||^2 \geq 2p||\nabla(\ln f)||^2, \]

where \( \nabla \ln f \) is the gradient of \( \ln f \) on \( N_K \) and \( p = \dim N_\perp \).

If the equality sign of (9.2) holds identically, then \( N_K \) is a totally geodesic Kähler submanifold and \( N_\perp \) is a totally umbilical totally real submanifold of \( \tilde{M} \). Moreover, \( N_K \times_f N_\perp \) is minimal in \( \tilde{M} \).
When $M$ is anti-holomorphic, i.e., when $J\mathbf{D}^\perp_x = T^\perp_x N$, and $p > 1$. The equality sign of (9.2) holds identically if and only if $N_\perp$ is a totally umbilical submanifold of $\tilde{M}$.

If $M$ is anti-holomorphic and $p = 1$, then the equality sign of (9.2) holds identically if and only if the characteristic vector field $J\xi$ of $M$ is a principal vector field with zero as its principal curvature. (Notice that in this case, $M$ is a real hypersurface in $\tilde{M}$.) Also, in this case, the equality sign of (9.2) holds identically if and only if $M$ is a minimal hypersurface in $\tilde{M}$.

Remark 9.1. Theorem 9.1 and Theorem 9.3 have been extended to twisted product $CR$-submanifolds in [16].

$CR$-warped products in complex space forms satisfying the equality case of (9.2) have been completely classified. In fact, we have the following.

Theorem 9.4. ([19]) A $CR$-warped product $N_K \times f N_\perp$ in $\mathbb{C}^m$ satisfies

$$||\sigma||^2 = 2p||\nabla (\ln f)||^2,$$

(9.3)

if and only if

(i) $N_K$ is an open portion of a complex Euclidean $h$-space $\mathbb{C}^h$,

(ii) $N_\perp$ is an open portion of the unit $p$-sphere $S^p$,

(iii) there is $a = (a_1, \ldots, a_h) \in S^{h-1} \subset \mathbb{H}^h$ such that $f = \sqrt{(a, z)^2 + (i a, z)^2}$ for $z = (z_1, \ldots, z_h) \in \mathbb{C}^h$, $w = (w_0, \ldots, w_p) \in S^p \subset \mathbb{H}^{p+1}$, and

(iv) up to rigid motions, the immersion is given by

$$x(z, w) = \left( z_1 + (w_0 - 1)a_1 \sum_{j=1}^h a_j z_j, \ldots, z_h + a_h \sum_{j=1}^h a_j z_j, w_1 \sum_{j=1}^h a_j z_j, \ldots, w_p \sum_{j=1}^h a_j z_j, 0, \cdots, 0 \right).$$

(9.4)

A $CR$-warped product $N_T \times f N_\perp$ is said to be trivial if its warping function $f$ is constant. A trivial $CR$-warped product $N_T \times f N_\perp$ is nothing but a $CR$-product $N_T \times N^f_\perp$, where $N^f_\perp$ is the manifold with metric $f^2 g_{N_\perp}$ which is homothetic to the original metric $g_{N_\perp}$ on $N_\perp$.

The following result classifies $CR$-warped products in complex projective spaces satisfying the equality case of (9.2).
Theorem 9.5. ([20]) A non-trivial CR-warped product $N_T \times f N_\perp$ in $C P^m(4)$ satisfies the basic equality $||\sigma||^2 = 2p||\nabla(\ln f)||^2$ if and only if we have

1. $N_T$ is an open portion of complex Euclidean $h$-space $C^h$,
2. $N_\perp$ is an open portion of a unit $p$-sphere $S^p$, and
3. up to rigid motions, the immersion $x$ of $N_T \times f N_\perp$ into $C P^m(4)$ is the composition $\pi \circ \bar{x}$, where

$$\bar{x}(z, w) = \left( z_0 + (w_0 - 1)a_0 \sum_{j=0}^{h} a_j z_j, \cdots, z_h + (w_0 - 1)a_h \sum_{j=0}^{h} a_j z_j, \right.$$  

$$w_1 \sum_{j=0}^{h} a_j z_j, \cdots, w_p \sum_{j=0}^{h} a_j z_j, 0, \ldots, 0 \right),$$  

(9.5)

$\pi$ is the projection $\pi : C_0^{m+1} \rightarrow C P^m(4)$, $a_0, \ldots, a_h$ are real numbers satisfying $a_0^2 + a_1^2 + \cdots + a_h^2 = 1$, $z = (z_0, z_1, \ldots, z_h) \in C^{h+1}$ and $w = (w_0, \ldots, w_p) \in S^p \subset \mathbb{H}^{p+1}$.

Similarly, we have the following classification theorems for CR-warped products in complex hyperbolic spaces satisfying the equality case of (9.2).

Theorem 9.6. ([20]) A CR-warped product $N_T \times f N_\perp$ in $C H^m(-4)$ satisfies the basic equality $||\sigma||^2 = 2p||\nabla(\ln f)||^2$ if and only if one of the following two cases occurs:

1. $N_T$ is an open portion of complex Euclidean $h$-space $C^h$, $N_\perp$ is an open portion of a unit $p$-sphere $S^p$ and, up to rigid motions, the immersion is the composition $\pi \circ \bar{x}$, where $\pi$ is the projection $\pi : C_0^{m+1} \rightarrow C H^m(-4)$ and

$$\bar{x}(z, w) = \left( z_0 + a_0(1-w_0) \sum_{j=0}^{h} a_j z_j, z_1 + a_1(w_0-1) \sum_{j=0}^{h} a_j z_j, \cdots, z_h + a_h(w_0-1) \sum_{j=0}^{h} a_j z_j, \right.$$  

$$z_1 + a_1(w_0-1) \sum_{j=0}^{h} a_j z_j, \cdots, z_h + a_h(w_0-1) \sum_{j=0}^{h} a_j z_j, \right),$$  

(9.6)

for some real numbers $a_0, \ldots, a_h$ satisfying $a_0^2 - a_1^2 - \cdots - a_h^2 = -1$, where $z = (z_0, \ldots, z_h) \in C^{h+1}$ and $w = (w_0, \ldots, w_p) \in S^p \subset \mathbb{H}^{p+1}$.

2. $p = 1$, $N_T$ is an open portion of $C^h$ and, up to rigid motions, the immersion
is the composition $\pi \circ \tilde{x}$, where

$$\tilde{x}(z, t) = \left( z_0 + a_0(\cosh t - 1) \sum_{j=0}^{h} a_j z_j, z_1 + a_1(1 - \cosh t) \sum_{j=0}^{h} a_j z_j, \ldots, z_h + a_h(1 - \cosh t) \sum_{j=0}^{h} a_j z_j, \sinh t \sum_{j=0}^{h} a_j z_j, 0, \ldots, 0 \right), \quad (9.7)$$

for some real numbers $a_0, a_1, \ldots, a_{h+1}$ satisfying $a_0^2 - a_1^2 - \cdots - a_{h}^2 = 1$.

10. Another Solution to Problem 9.1

In fact, $CR$-warped products in a complex space form also satisfy another general inequality; which provides the second solution to Problem 9.1.

**Theorem 10.1.** ([26]) Let $N = N^h_K \times_f N^p_\perp$ be a $CR$-warped product in a complex space form $\tilde{M}(4c)$ of constant holomorphic sectional curvature $c$. Then we have

$$||\sigma||^2 \geq 2p(||\nabla(\ln f)||^2 + \Delta(\ln f) + 2hc). \quad (10.1)$$

If the equality sign of (10.1) holds identically, then $N_K$ is a totally geodesic submanifold and $N_\perp$ is a totally umbilical submanifold. Moreover, $N$ is a minimal submanifold in $\tilde{M}(4c)$.

One may also classify $CR$-warped products which satisfy the equality case of (10.1) as given in the following theorems.

**Theorem 10.2.** ([26]) Let $\phi : N^h_K \times_f N^p_\perp \to \mathbb{C}^m$ be a $CR$-warped product in $\mathbb{C}^m$. Then we have

$$||\sigma||^2 \geq 2p\{||\nabla(\ln f)||^2 + \Delta(\ln f)\}. \quad (10.2)$$

The equality case of (10.2) holds if and only if

(a) $N_K$ is an open portion of $\mathbb{C}^h$ = $\mathbb{C}^h - \{0\}$;
(b) $N_\perp$ is an open portion of $S^p$;
(c) There is $\alpha$, $1 \leq \alpha \leq h$, and complex Euclidean coordinates $\{z_1, \ldots, z_h\}$ on $\mathbb{C}^h$ such that $f = \sqrt{\sum_{j=1}^{\alpha} z_j \bar{z}_j}$;
Up to rigid motions, the immersion $\phi$ is given by

$$\phi = (w_0 z_1, \ldots, w_p z_1, \ldots, w_0 z_\alpha, \ldots, w_p z_\alpha, z_{\alpha+1}, \ldots, z_h, 0, \ldots, 0),$$

for $z = (z_1, \ldots, z_h) \in \mathbb{C}_h^h$ and $w = (w_0, \ldots, w_p) \in \mathbb{S}^p \subset \mathbb{E}^{p+1}$.

**Theorem 10.3.** ([26]) Let $\phi : N_T \times f N_\bot \rightarrow CP^m(4)$ be a CR-warped product with $\dim_{\mathbb{C}} N_T = h$ and $\dim_{\mathbb{R}} N_\bot = p$. Then we have

$$||\sigma||^2 \geq 2p(||\nabla (\ln f)||^2 + \Delta (\ln f) + 2h).$$

(10.4)

The CR-warped product satisfies the equality case of (10.4) if and only if

(i) $N_T$ is an open portion of complex projective $h$-space $CP^h(4)$;

(ii) $N_\bot$ is an open portion of unit $p$-sphere $S^p$; and

(iii) There exists a natural number $\alpha \leq h$ such that, up to rigid motions, $\phi$ is the composition $\pi \circ \tilde{\phi}$, where

$$\tilde{\phi}(z, w) = (w_0 z_0, \ldots, w_p z_0, \ldots, w_0 z_\alpha, \ldots, w_p z_\alpha, z_{\alpha+1}, \ldots, z_h, 0, \ldots, 0),$$

for $z = (z_0, \ldots, z_h) \in \mathbb{C}_h^{h+1}$ and $w = (w_0, \ldots, w_p) \in \mathbb{S}^p \subset \mathbb{E}^{p+1}$, where $\pi$ is the projection $\pi : \mathbb{C}_h^{m+1} \rightarrow CP^m(4)$.

**Theorem 10.4.** ([26]) Let $\phi : N_T \times f N_\bot \rightarrow CH^m(-4)$ be a CR-warped product with $\dim_{\mathbb{C}} N_T = h$ and $\dim_{\mathbb{R}} N_\bot = p$. Then we have

$$||\sigma||^2 \geq 2p(||\nabla (\ln f)||^2 + \Delta (\ln f) - 2h).$$

(10.6)

The CR-warped product satisfies the equality case of (10.6) if and only if

(a) $N_T$ is an open portion of complex hyperbolic $h$-space $CH^h(-4)$;

(b) $N_\bot$ is an open portion of unit $p$-sphere $S^p$ (or $\mathbb{R}$, when $p = 1$); and

(c) up to rigid motions, $\phi$ is the composition $\pi \circ \tilde{\phi}$, where either $\tilde{\phi}$ is given by

$$\tilde{\phi}(z, w) = (z_0, \ldots, z_\beta, w_0 z_\beta, \ldots, w_p z_\beta, \ldots, w_0 z_h, \ldots, w_p z_h, 0, \ldots, 0),$$

for $0 < \beta \leq h$, $z = (z_0, \ldots, z_h) \in \mathbb{C}_h^{h+1}$ and $w = (w_0, \ldots, w_p) \in \mathbb{S}^p$, or $\tilde{\phi}$ is given by

$$\tilde{\phi}(z, u) = (z_0 \cosh u, z_0 \sinh u, z_1 \cos u, z_1 \sin u, \ldots, \ldots, z_\alpha \cos u, z_\alpha \sin u, z_{\alpha+1}, \ldots, z_h, 0 \ldots, 0),$$

(10.7)
for \( z = (z_0, \ldots, z_h) \in \mathbb{C}^{h+1} \) and \( u \in \mathbb{R} \), where \( \pi \) is the projection \( \pi : \mathbb{C}^{m+1} \to CH^m(-4) \).

11. Convolution of Riemannian Manifolds

Convolution of Riemannian manifolds introduced in [23, 29] is a natural extension of direct and warped products. Convolution of Riemannian manifolds is defined as follows:

Let \((N_1, g_1)\) and \((N_2, g_2)\) be two Riemannian manifolds and \( f \) and \( h \) be positive functions on \( N_1 \) and \( N_2 \), respectively. Consider the symmetric tensor field \( g_{f,h} \) of type \((0,2)\) on \( N_1 \times N_2 \) defined by

\[
h g_1 \ast f g_2 = h^2 g_1 + f^2 g_2 + 2fhdf \otimes dh,
\]

which is called the convolution of \( g_1 \) and \( g_2 \) via \( h \) and \( f \). The product manifold \( N_1 \times N_2 \) equipped with \( h g_1 \ast f g_2 \) is called a convolution manifold, denoted by \( h N_1 \ast f N_2 \).

When the scale functions \( f, h \) are irrelevant, we simply denote \( h N_1 \ast f N_2 \) and \( h g_1 \ast f g_2 \) by \( N_1 \ast f N_2 \) and \( g_1 \ast g_2 \), respectively.

If \( h g_1 \ast f g_2 \) is nondegenerate, it defines a pseudo-Riemannian metric on \( N_1 \times N_2 \) with index \( \leq 1 \). In this case, \( h g_1 \ast f g_2 \) is called a convolution metric and \( h N_1 \ast f N_2 \) is called a convolution pseudo-Riemannian manifold. If the index of the pseudo-Riemannian metric is zero, \( h N_1 \ast f N_2 \) is called a convolution Riemannian manifold.

The notion of convolution arises naturally, since each tensor product immersion gives rise to a convolution as shown in the next theorem.

**Theorem 11.1.** ([23]) Let \( \psi_1 : (N_1, g_1) \to \mathbb{E}_*^n \subset \mathbb{E}^n \) and \( \psi_2 : (N_2, g_2) \to \mathbb{E}_*^m \subset \mathbb{E}^m \) be isometric immersions, where \( \mathbb{E}_*^n = \mathbb{E}^n - \{0\} \). Then

\[
\psi = \psi_1 \otimes \psi_2 : N_1 \times N_2 \to \mathbb{E}^n \otimes \mathbb{E}^m = \mathbb{E}^{nm} ; \quad (u, v) \mapsto \psi_1(u) \otimes \psi_2(v),
\]

(11.2)
gives rise to a convolution manifold \( N_1 \ast f N_2 \) equipped with

\[
\rho_2 g_1 \ast \rho_1 g_2 = \rho_2^2 g_1 + \rho_1^2 g_2 + 2\rho_1 \rho_2 \rho_1 \otimes \rho_2,
\]

(11.3)
where \( \rho_1 \) and \( \rho_2 \) are the distance functions of \( \psi_1 \) and \( \psi_2 \), respectively.
Therefore, we have the following induced metrics:

\[ \phi_1 \oplus \phi_2 : N_1 \times N_2 \to E^{m_1} \oplus E^{m_2} \implies g_1 + g_2 \text{ (direct sum)} \]

\[ \psi_1 \otimes \psi_2 : N_1 \times N_2 \to E^n \otimes E^m \implies \rho_2 g_1 \ast_{\rho_1} g_2 \text{ (convolution)} \]

**Example 11.1.** If \( \psi_2 : (N_2, g_2) \to E^n \) is an isometric immersion such that \( \psi_2(N_2) \) is contained in \( S^{m-1}(1) \subset E^m \) which is centered at the origin. Then the convolution \( \rho_2 g_1 \ast_{\rho_1} g_2 \) on \( N_1 \star N_2 \) is the warped product metric: \( g_1 + \rho_2^2 g_2 \).

**Definition 11.1.** A convolution \( h g_1 \ast f g_2 \) of two Riemannian metrics \( g_1 \) and \( g_2 \) is called degenerate if \( \det(h g_1 \ast f g_2) = 0 \) holds identically.

For \( X \in T(N_j) \), \( j = 1, 2 \), we denote by \( |X|_j \) the length of \( X \) with respect to metric \( g_j \) on \( N_j \).

The following theorem determines when a convolution is degenerate.

**Theorem 11.2.** ([23]) Let \( h N_1 \star f N_2 \) be the convolution of Riemannian manifolds \( (N_1, g_1) \) and \( (N_2, g_2) \) via \( h \) and \( f \). Then \( h g_1 \ast f g_2 \) is degenerate if and only if we have

1. the length \( |\text{grad} f|_1 \) of the gradient of \( f \) on \( (N_1, g_1) \) is a nonzero constant, say \( c \), and
2. the length \( |\text{grad} h|_2 \) of the gradient of \( h \) on \( (N_2, g_2) \) is the constant given by \( 1/c \), i.e., the reciprocal of \( c \).

The following result provides a necessary and sufficient condition for \( h g_1 \ast f g_2 \) to be Riemannian.

**Theorem 11.3.** ([23]) Let \( h N_1 \star f N_2 \) be the convolution of Riemannian manifolds \( (N_1, g_1) \) and \( (N_2, g_2) \) via \( h \) and \( f \). Then \( h g_1 \ast f g_2 \) is Riemannian if and only if we have \( |\text{grad} f|_1 |\text{grad} h|_2 < 1 \).

In views of Theorems 11.2 and 11.3, it is natural to study Euclidean submanifolds whose distance function \( \rho \) satisfies the condition: \( |\text{grad} \rho| = \text{constant} \). To do so, we introduce the notion of constant-ratio submanifolds.

**Definition 11.2.** Let \( x : M \to E^n \) be an isometric immersion. We put

\[ x = x^N + x^T, \quad (11.4) \]
where $x^N$ and $x^T$ are the normal and tangential components of the position vector field, respectively. The submanifold $M$ is said to be of constant-ratio if the ratio $|x^N| : |x^T|$ is constant; or equivalently, $|x^T| : |x| = c = \text{constant}$ (cf. [21]).

The following results characterize constant-ratio submanifolds.

**Theorem 11.4.** ([23]) Let $x : M \to \mathbb{E}^n$ be an isometric immersion. Then $M$ is of constant-ratio with $|x^T| : |x| = c$ if and only if $|\text{grad} \rho| = c$ which is constant.

In particular, for submanifold of constant-ratio, we have $|\text{grad} \rho| = c \leq 1$.

Theorem 11.3 and Theorem 11.4 imply

**Corollary 11.1.** Let $\psi_1 : (N_1, g_1) \to \mathbb{E}^n$ and $\psi_2 : (N_2, g_2) \to \mathbb{E}^m$ be isometric immersions and $\rho_1 = |\psi_1|$ and $\rho_2 = |\psi_2|$ be their distance functions. Then $\rho_2 g_1 * \rho_1 g_2$ is degenerate if and only if $|\text{grad} \rho_1| = |\text{grad} \rho_2| = 1$.

Space-like constant-ratio submanifolds in pseudo-Euclidean space have been completely classified in [25]. In particular, for constant-ratio submanifolds in Euclidean space, we have the following.

**Theorem 11.5.** ([25]) An isometric immersion $x : M \to \mathbb{E}^m$ is of constant-ratio if and only if one of the following three cases occurs:

1. $x(M)$ is contained in hypercone with vertex at the origin.
2. $x(M)$ is contained in a hypersphere $S^{m-1}(r^2)$ of $\mathbb{E}^m$ centered at the origin.
3. There is a number $b \in (0,1)$ and local coordinate system $\{s, u_2, \ldots, u_n\}$ on $M$ such that the immersion $x$ is given by

$$x(s, u_2, \ldots, u_n) = bsY(s, u_2, \ldots, u_n), \quad (11.5)$$

where $Y = Y(s, u_2, \ldots, u_n)$ satisfies conditions:

3.a) $Y = Y(s, u_2, \ldots, u_n)$ lies in $S^{m-1}(1)$,

3.b) $Y_s$ is perpendicular to $Y_{u_2}, \ldots, Y_{u_n}$, and

3.c) $|Y_s| = \sqrt{1 - b^2/(bs)}$.

**Remark 11.1.** Submanifolds of constant-ratio also relate to a problem in physics concerning the motion in a central force field which obeys the inverse-cube law. For example, the trajectory of a particle subject to a central force of
attraction located at the origin which obeys the inverse-cube law is a curve of constant-ratio.

**Remark 11.2.** The inverse-cube law was originated from Sir Isaac Newton in his letter sent to Robert Hooke on December 13, 1679. This letter is of great historical importance because it reveals the state of Newton’s development of dynamics at that time (cf. [45, 46]).

### 12. Applications of Convolution

We define an isometric map from a convolution manifold into a Riemannian manifold as follows.

**Definition 12.1.** A map \( \psi : (\mathcal{N}_1 \star \mathcal{N}_2, h g_1 \ast f g_2) \to (\mathcal{M}, \bar{g}) \) from a convolution manifold into a Riemannian manifold is called isometric if \( \psi^* \bar{g} = h g_1 \ast f g_2 \).

For example, let \( z : \mathbb{C}^h_x \to \mathbb{C}^h \) and \( x : \mathbb{E}^p \to \mathbb{E}^p \) be inclusion maps. Consider the map

\[
\psi_{z,x} = z \otimes x : \mathbb{C}^h_x \times \mathbb{E}^p \to (\mathbb{C}^{hp}, \tilde{g}_0),
\]

(12.1)

defined by

\[
\psi_{z,x} = z \otimes x = (z_1 x_1, \ldots, z_1 x_p, \ldots, z_h x_1, \ldots, z_h x_p),
\]

(12.2)

for \( z = (z_1, \ldots, z_h) \in \mathbb{C}^h_x \) and \( x = (x_1, \ldots, x_p) \in \mathbb{E}^p \). Then we have

\[
\psi_{z,x}^* (\tilde{g}_0) = \rho_2 g_1 \ast \rho_1 g_2 = \rho_2^2 g_1 + \rho_1^2 g_2 + 2 \rho_1 \rho_2 d \rho_1 \otimes d \rho_2,
\]

(12.3)

where \( g_1, g_2 \) and \( \rho_1, \rho_2 \) are the metrics and distance functions of \( \mathbb{C}^h_x, \mathbb{E}^p \), respectively. Thus

\[
\psi_{z,x} : (\mathbb{C}^h_x \times \mathbb{E}^p, \rho_2 g_1 \ast \rho_1 g_2) \to (\mathbb{C}^{hp}, \tilde{g}_0)
\]

(12.4)

is an isometric map.

An isometric map

\[
\phi : (\mathbb{C}^h_x \times \mathbb{E}^p, \rho_2 g_1 \ast \rho_1 g_2) \to (\mathbb{C}^m, \bar{g}_0)
\]

(12.5)

is call a CR-map if \( \phi \) carries each slice \( \mathbb{C}^h_x \times \{v\} \) of \( \mathbb{C}^h_x \times \mathbb{E}^p \) into a Kähler submanifold and each slice \( \{u\} \times \mathbb{E}^p \) of \( \mathbb{C}^h_x \times \mathbb{E}^p \) into a totally real submanifold.
The notion of convolution is useful in the study of tensor product immersions. For instance, we may applying the notion of convolution to obtain the following two simple characterizations of the map \( \psi_{z,x} = z \otimes x : \mathbb{C}^h \times \mathbb{E}_s^p \to \mathbb{C}^{hp} \) defined by (12.2).

**Theorem 12.1.** Let \( \phi : U \to \mathbb{C}^m \) be an isometric CR-map from an open portion \( U \) of the convolution manifold \( (\mathbb{C}^h \times \mathbb{E}_s^p, \rho_2 g_1 \ast \rho_1 g_2) \) into complex Euclidean \( m \)-space. Then we have

1. \( m \geq hp \).
2. If \( m = hp \), then, up to rigid motions of \( \mathbb{C}^m \), \( \phi \) is the map \( \psi_{z,x} = z \otimes x \) defined by (12.2).

**Theorem 12.2.** Let \( \phi : U \to \mathbb{C}^m \) be an isometric CR-map from an open portion \( U \) of the convolution manifold \( (\mathbb{C}^h \times \mathbb{E}_s^p, \rho_2 g_1 \ast \rho_1 g_2) \) into a complex Euclidean space. Then the second fundamental form \( \sigma \) of \( \phi \) satisfies

\[
||\sigma||^2 \geq \frac{(2h-1)(p-1)}{\rho_2^2 \rho_2^2}.
\]

(12.6)

The equality sign of (12.6) holds identically if and only if, up to rigid motions of \( \mathbb{C}^m \), \( \phi \) is given by \( \phi(z, x) = (z \otimes x, 0, \ldots, 0) \).

These two theorems can be regarded as Euclidean version of Theorems 7.1 (with \( s = 2 \)) and 7.3 related with the converse to Segre imbedding.

13. T-Constant and N-Constant Submanifolds

In views of constant-ratio submanifolds in Euclidean space, it is natural to study submanifolds in Euclidean space such that either \( x^T \) or \( x^N \) has constant length, where \( x^T \) and \( x^N \) are the tangential and the normal components of the position function given in (11.4). For simplicity, we call a submanifold \( M \) in a Euclidean space a T-constant (respectively, N-constant) submanifold if \( x^T \) is of constant length (respectively, \( |x^N| \) is of constant length).

For T-constant submanifolds, we have the following.

**Theorem 13.1.** Let \( x : M \to \mathbb{E}^m \) be an isometric immersion of a Riemannian \( n \)-manifold into the Euclidean \( m \)-space. Then \( M \) is a T-constant submanifold if and only if either
(i) $x(M)$ is contained in a hypersphere $S^{m-1}(r^2)$ of $\mathbb{E}^m$, or

(ii) there exist a real numbers $a, b$ and local coordinate systems $\{s, u_2, \ldots, u_n\}$ on $M$ such that the immersion $x$ is given by

$$x(s, u_2, \ldots, u_n) = \sqrt{a^2 + b + 2as} Y(s, u_2, \ldots, u_n),$$

where $Y = Y(s, u_2, \ldots, u_n)$ satisfies conditions:

(2.a) $Y = Y(s, u_2, \ldots, u_n)$ lies in the unit hypersphere $S^{m-1}(1)$,

(2.b) the coordinate vector field $Y_s$ is perpendicular to coordinate vector fields $Y_{u_2}, \ldots, Y_{u_n}$, and

(2.c) $Y_s$ satisfies $|Y_s| = \sqrt{b + 2as}/(a^2 + b + 2as)$.

For $N$-constant submanifolds, we have the following.

**Theorem 13.2.** ([30]) Let $x : M \to \mathbb{E}^m$ be an isometric immersion of a Riemannian $n$-manifold into the Euclidean $m$-space. Then $M$ is $N$-constant if and only if one of the following three cases occurs:

(1) $x$ is a conic submanifold with the vertex at the origin.

(2) $x(M)$ is contained in a hypersphere $S^{m-1}(r^2)$ of $\mathbb{E}^m$.

(3) There exist a positive number $c$ and local coordinate systems $\{s, u_2, \ldots, u_n\}$ on $M$ such that the immersion $x$ is given by

$$x(s, u_2, \ldots, u_n) = \sqrt{s^2 + c^2} Y(s, u_2, \ldots, u_n),$$

where $Y = Y(s, u_2, \ldots, u_n)$ satisfies conditions:

(3.a) $Y = Y(s, u_2, \ldots, u_n)$ lies in the unit hypersphere $S^{m-1}(1)$,

(3.b) the coordinate vector field $Y_s$ is perpendicular to coordinate vector fields $Y_{u_2}, \ldots, Y_{u_n}$, and

(3.c) $Y_s$ satisfies $|Y_s| = c/(s^2 + c^2)$.

14. Rectifying curves and $\tau/\kappa$

Consider a unit-speed space curve $x : I \to \mathbb{R}^3$, $I = (\alpha, \beta)$, in $\mathbb{E}^3$ that has at least four continuous derivatives. Let $t$ denote $x'$. It is possible that $t'(s) = 0$ for some $s$, however, we assume that this never happens. Then we can introduce a unique vector field $n$ and positive function $\kappa$ so that $t' = \kappa n$. We call $t'$ the
curvature vector field, \( n \) the *principal normal vector field*, and \( \kappa \) the *curvature*. Since \( t \) is a constant length vector field, \( n \) is orthogonal to \( t \). The *binormal vector field* is defined by \( b = t \times n \) which is a unit vector field orthogonal to both \( t \) and \( n \). One defines the *torsion* \( \tau \) by the equation \( b' = -\tau n \).

The famous Serret-Frenet equations are given by
\[
\begin{align*}
    t' &= \kappa n, \\
    n' &= -\kappa t + \tau b, \quad \text{(14.1)} \\
    b' &= -\tau n.
\end{align*}
\]

At each point of the curve, the planes spanned by \( \{t, n\} \), \( \{t, b\} \) and \( \{n, b\} \) are known as the *osculating plane*, the *rectifying plane* and the *normal plane*, respectively. A curve in \( \mathbb{E}^3 \) is called a *twisted curve* if has nonzero curvature and torsion.

In elementary differential geometry it is well-known that a curve in \( \mathbb{E}^3 \) is a planar curve if its position vector lies in its osculating plane at each point. And it is a spherical curve if its position vector lies in its normal plane at each point. In views of these basic facts, it is natural to ask the following simple geometric question:

**Problem 14.1.** *When the position vector of a space curve \( x : I \to \mathbb{E}^3 \) always lies in its rectifying plane?*

For simplicity we call such a curve a *rectifying curve*. So, for a rectifying curve \( x : I \to \mathbb{E}^3 \), the position vector \( x \) satisfies
\[
x(s) = \lambda(s)t(s) + \mu(s)b(s), \quad \text{(14.2)}
\]
for some functions \( \lambda \) and \( \mu \).

The following result provides some simple characterizations of rectifying curves. In particular, it shows that rectifying curves in \( \mathbb{E}^3 \) are \( N \)-constant curves.

**Theorem 14.1.** ([31]) *Let \( x : I \to \mathbb{E}^3 \) be a rectifying curve in \( \mathbb{E}^3 \) with \( \kappa > 0 \) and \( s \) be its arclength function. Then we have*

(i) *The curve is \( N \)-constant and the distance function \( \rho \) is non-constant.*
(ii) The distance function \( \rho = |\mathbf{x}| \) satisfies \( \rho^2 = s^2 + c_1 s + c_2 \) for some constants \( c_1 \) and \( c_2 \).

(iii) The tangential component of the position vector of the curve is given by \( \langle \mathbf{x}, \mathbf{t} \rangle = s + c \) for some constant \( c \).

(iv) the torsion \( \tau \) is non-zero and the binormal component of the position vector is constant, i.e. \( \langle \mathbf{x}, \mathbf{b} \rangle \) is constant.

Conversely, if \( \mathbf{x} : I \to \mathbb{E}^3 \) is a curve with \( \kappa > 0 \) and one of (i), (ii), (iii) or (iv) holds, then \( \mathbf{x} \) is a rectifying curve.

The fundamental existence and uniqueness theorem for curves in \( \mathbb{E}^3 \) states that a curve is uniquely determined up to rigid motions by its curvature and torsion, given as functions of its arclength. Furthermore, given continuous functions \( \kappa(s) \) and \( \tau(s) \), with \( \kappa(s) \) positive and continuously differentiable, there is a differentiable curve (of class at least \( C^3 \)) with curvature \( \kappa \) and torsion \( \tau \). In general, a result of Lie and Darboux shows that solving Serret-Frenet equations is equivalent to solving a certain complex Riccati equation (see [53, p.36]). In practice, space curves with prescribed curvature and torsion are often impossible to find explicitly. Fortunately, we are able to determine completely all rectifying curves in \( \mathbb{E}^3 \).

**Theorem 14.2.** ([31]) Let \( \mathbf{x} : I \to \mathbb{E}^3 \) be a curve \( \mathbb{E}^3 \) with \( \kappa > 0 \). Then it is a rectifying curve if and only if, up to parameterization, it is given by

\[
\mathbf{x}(t) = a(\sec t)\mathbf{y}(t),
\]

where \( a \) is a positive number and \( \mathbf{y} = \mathbf{y}(t) \) is a unit speed curve in \( S^2 \), where \( S^2 \) denotes the unit sphere in \( \mathbb{E}^3 \) centered at the origin.

A generalized helix in Euclidean 3-space is a twisted curve whose unit tangent vector field \( \mathbf{t} \) makes constant angle with a fixed direction. It is well-known that a twisted curve is a generalized helix if and only if the ratio \( \tau/\kappa \) is a non-zero constant. So, it is natural to ask the following.

**Problem 14.2.** When the ratio \( \tau/\kappa \) of a twisted curve is a non-constant linear function in arclength function?

The following result provides an answer to this Problem.
Theorem 14.3. (31) Let \( x : I \rightarrow \mathbb{E}^3 \) be a twisted curve in \( \mathbb{E}^3 \). Then \( x \) is congruent to a rectifying curve if and only if the ratio of torsion and curvature of the curve is a non-constant linear function in arclength function \( s \), i.e., \( \tau/\kappa = c_1s + c_2 \) for some constants \( c_1, c_2 \) with \( c_1 \neq 0 \).

Theorem 14.3 seems to be of independent interest by itself.

References


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