A UNIFIED APPROACH TO UNIQUENESS, EXPANSION, APPROXIMATION, AND INTERPOLATION PROBLEMS

BY

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Abstract. Using the concept of Pólya property, this paper solves the interpolation problem by proving that, given a sequence of complex numbers \( \{b_n\} \), there exists \( f \) in \( K[\Omega] \) such that \( L_n(f) = b_n, \ n = 0, 1, \ldots \), where \( K[\Omega] \) is a class of entire functions of exponential type whose Borel transforms are analytic on \( \Omega \) and \( L_n \) is a sequence of continuous linear functionals defined on \( K[\Omega] \) with generating kernels \( g_n, \ n = 0, 1, \ldots \). Combining this with the author’s research monograph “A Unified Approach to Uniqueness, Expansion and Approximation Problems”, the author has finally used the same approach to solve four areas of problems.

1. Introduction

In the papers [4, 5, 8] and the research monograph [6], the author used a concept called the Pólya property of a holomorphic function \( m(w, z) \) of two variables to study problems in the areas of (1) uniqueness classes for entire functions of exponential type, (2) expansion of entire functions of exponential type in interpolation series, and (3) approximation of functions analytic on a simply connected domain by linear combinations of functions \( \{g_n\}_{n=0}^\infty \). It turns out that the concept of Pólya property can also be used to solve the interpolation problem. The main part of this paper is devoted to show this result.
2. The Function Spaces

Let $\Omega$ be a simply connected domain. Then $H(\Omega)$ denotes the class of all functions analytic on $\Omega$. Let $K$ denote the class of all entire functions of exponential type [1]. Then $K[\Omega]$ denotes the class of $f$ in $K$ such that the Borel transform of $f$ is analytic on $\Omega^c$, the complement of $\Omega$ taken relative to the Riemann sphere. Let $BK[\Omega]$ denote the space of all Borel transforms of functions in $K[\Omega]$. Then $BK[\Omega]$ is the space of all functions analytic on $\Omega^c$ that vanish at $\infty$.

With appropriate topological structures being built into the above spaces ([6, p.4-8]), one can see clearly the relationship among the above spaces, using Köthe's duality theorem ([14]), among others. Let $\Omega_z$ and $\Omega_\eta$ be simply connected domains and $T$ a continuous linear operator from $H(\Omega_\eta)$ to $H(\Omega_z)$. Then $T'$, a continuous linear operator from $BK[\Omega_z]$ to $BK[\Omega_\eta]$, is the conjugate of $T$ ([6, p.23]). Let $B_Z : K[\Omega_z] \rightarrow BK[\Omega_z]$ be the operator which maps $f$ to $B(f)$ for each $f$ in $K[\Omega_z]$ and $\phi$ a continuous linear operator from $BK[\Omega_z]$ to $K[\Omega_\eta]$. Then $\phi \circ B_Z$ is a continuous linear operator from $K[\Omega_z]$ to $K[\Omega_\eta]$ ([6, p.24-25]).

The following diagram ([6, p.12]) clarifies the relationships among the above spaces and operators:

![Diagram]

3. The Pólya Property

Pólya’s Lemma ([1, p.110]) can be generalized obviously [6, p.20-22]. Let
Let $m(w, z)$ be holomorphic on $\Omega_w \times \Omega_z$. Then $m$ has the Pólya property with respect to $z$ on $\Omega_z$ if and only if for all simple closed contours $\Gamma \subseteq \Omega_z$, if $F$ is analytic outside and on $\Gamma$ with $F(\infty) = 0$ and if $\int_{\Gamma} m(w, z) F(z) dz \equiv 0$, then $F \equiv 0$. Due to Köthe’s representation theorems ([14]), the continuous linear operator $T$ from $H(\Omega_\eta)$ to $H(\Omega_z)$ and its conjugate $T'$ from $BK[\Omega_z]$ to $BK[\Omega_\eta]$ all have their integral representations ([6, p.23]) with the kernel function $M(\eta, z)$ locally holomorphic on $\Omega_\eta \times \Omega_z$.

It can also be proven ([6, p.24-25]) that the continuous linear operator $\phi o B_z$ from $K[\Omega_z]$ to $K[\Omega_\eta]$ has an integral representation with the kernel function $m(w, z)$ holomorphic on $C \times \Omega_z$. It is known ([6, p.25]) that $\phi o B_z$ is one-to-one if and only if its corresponding kernel function $m(w, z)$ has the Pólya property with respect to $z$ on $\Omega_z$. Similarly, the conjugate operator $T'$ from $BK[\Omega_z]$ to $BK[\Omega_\eta]$ is one-to-one if and only if its corresponding kernel function $M(\eta, z)$ has the Pólya property on $\Omega_z$.

4. The Pólya Property and Uniqueness

Let $\Omega$ be a simply connected domain in $C$. Let $\{L_n\}$ be a sequence of linear functionals defined on $K[\Omega]$ by

$$L_n(f) = \frac{1}{2\pi i} \int_{\Gamma} g_n(z) F(z) dz,$$

where $g_n$ is analytic on $\Omega$, $n = 0, 1, 2, \ldots$, and $F \subseteq \Omega$ is a simple closed contour so chosen that the Borel transform $F$ of $f$ is analytic outside and on $\Gamma$.

Let $\Omega_w$ be a domain and $\Omega_z$ a simply connected domain in $C$. Let $m(w, z)$ be holomorphic on $\Omega_w \times \Omega_z$. If $g_n(z)$ can be defined in terms of $m(w, z)$ in some way such as $g_n(z) = m(w_n, z)$ where $\{w_n\}$ is a sequence of points in $\Omega_w$ with a limit point in $\Omega_w$ and $g_n(z) = D_w^{(n)}[m(w, z)]_{w=w_0}$, $n = 0, 1, 2, \ldots$, where $D_w^{(n)}[m(w, z)]_{w=w_0}$ is the $n$-th partial derivative of $m(w, z)$ with respect to $w$ evaluated at $w_0$ ([4], [6, p.34-46]), then $m(w, z)$ has the Pólya property on $\Omega_z$ if and only if $K[\Omega_z]$ is a uniqueness class for the sequence $\{L_n\}$ of linear functionals defined on $K[\Omega_z]$ by (4.1).
5. The Pólya Property and Expansion

A sequence \( \{P_n\} \) of analytic functions (frequently polynomials) which satisfies a formal relation \( m(w, z) = \sum_{n=0}^{\infty} p_n(w)z^n \) is said to be generated by the kernel function \( m(w, z) \). A function \( f \) is said to have a unique expansion in a series of analytic functions \( p_n \) (unique polynomial expansion if each \( p_n \) is a polynomial) if \( f(z) = \sum_{n=0}^{\infty} c_n p_n(z) \) and \( \sum_{n=0}^{\infty} h_n p_n(z) = 0 \) implies \( h_n = 0 \) for \( n = 0, 1, 2, \ldots \).

To ease the task of explaining the relationship between the Pólya property and expansion problem, we illustrate with the Appell generating kernel \( A(z)e^{wz} \) in the formal generating relation (5.1)

\[
A(z)e^{wz} = \sum_{n=0}^{\infty} p_n(w)z^n, \quad A(0) \neq 0, 
\]

where \( \{p_n\} \) is a sequence of Appell polynomials.

Boas and Buck ([5]) show that for \( A(z) \) analytic on a disk \( D : |z| < r \), a function \( f \) in \( K[D] \) has a convergent expansion in a series of Appell polynomials

\[
f(z) = \sum_{n=0}^{\infty} c_n p_n(z) \quad \text{with} \quad c_n = \frac{1}{2\pi i} \int_{\Gamma} \frac{z^n}{A(z)} F(z) dz,
\]

where \( \Gamma \subseteq D \) is a simple closed contour with \( A(z) \neq 0 \) for \( z \) on \( \Gamma \) and \( F \), the Borel transform of \( f \), is analytic outside and on \( \Gamma \). They show that the expansion is unique if and only if \( A \) has no zero in \( D \).

Let \( m(w, z) = A(z)e^{wz} \). Since we can show that \( m(w, z) \) has the Pólya property on \( D \) if and only if \( A(z) \) has no zero in \( D \), we obtain at once that each function in \( K[D] \) has a unique expansion in a series of Appell polynomials generated by (5.1) if and only if the generating kernel of (5.1) has the Pólya property on \( D \). This observation not only relates functions which have the Pólya property to expansion problem but also suggests that we may use functions \( m(w, z) \) which have the Pólya property to serve as generating kernels in formal generating relations and thus obtain new expansion results.

The author ([5], [6, p.47-67]) has indeed obtained numerous new expansion results, using the above approach. This new method has a few advantages over other methods:
1. It not only gives a unique expansion for a function but also a precise condition on the growth of its coefficients.
2. It provides new insight into the problem of finding causes of multiple expansions of entire functions. The cause of multiple expansions in our approach lies in the kernel function not having the Pólya property while the cause of multiple expansion in other approaches is the type of contour chosen.
3. When applying our method to existing generating kernels, we obtain at least the same expansion results but in a much simpler way.
4. We can use Mittag-Leffler summability, instead of convergence, when applying our method, and thus obtain a stronger result.

6. The Pólya Property and Approximation

In [8], [6, p.68-71], the author proves a theorem which shows that the uniqueness problem for entire functions of exponential type is equivalent to the approximation problem for analytic functions. Thus each uniqueness theorem for entire functions of exponential type corresponds an approximation theorem for analytic functions and conversely. This result is then combined with the known uniqueness theorems [4, 9, 10, 11, 12, 15] to yield many approximation theorems. Using examples of functions which are known to have the Pólya property ([6, 8, 9]), we are able to give more specific results in approximation.

7. The Pólya Property and Interpolation

Let \( \{L_n\} \) be a sequence of linear functionals defined on \( K[\Omega] \) with the generating kernel functions \( \{g_n\} \), \( n = 0, 1, 2, \ldots, \) as in (4.1). A sequence of complex numbers \( \{b_n\} \) is said to be interpolated by \( K[\Omega] \) relative to \( \{g_n\} \) if there exists \( f \) in \( K[\Omega] \) such that \( L_n(f) = b_n, \ n = 0, 1, 2, \ldots. \) Let \( T \) be a continuous linear operator from \( H(\Omega_\eta) \) to \( H(\Omega_z) \) as in the diagram of Section 2. Then \( T \) is said to be uniqueness preserving if \( K[\Omega_z] \) is a uniqueness class for \( \{T(g_n)\} \) whenever \( K[\Omega_\eta] \) is a uniqueness class for \( \{g_n\} \). The operator \( T \) is said to be interpolation preserving if \( K[\Omega_z] \) interpolates \( \{b_n\} \) relative to \( \{T(g_n)\} \) whenever \( K[\Omega_\eta] \) interpolates \( \{b_n\} \) relative to \( \{g_n\} \).
Since the operator $T$ and its conjugate $T'$ are both one-to-one if and only if their respective kernel functions have the Polya property on $\Omega_z$ ([6, p.25-27]), it follows from the fact that both $H(\Omega_{\mu})$ and $H(\Omega_z)$ are Frechet spaces ([6, p.5]) and a theorem of Edwards [13, p.521-522] that $T$ and $T'$ are both onto and both homeomorphism if and only if the corresponding kernel functions have the Polya property on $\Omega_z$. By a theorem of Child ([9]), the operator $T$ is both uniqueness preserving and interpolation preserving if and only if the kernel functions of $T$ and $T'$ have the Polya property on $\Omega_z$. For the following theorem, a univalent function is defined as an analytic function which is one to one.

**Theorem 7.1.** Let $W$ be analytic and univalent on a simply connected domain $\Omega$. Let $\{b_n\}$ be a sequence of complex numbers and define a function $B$ by $B(z) = \sum_{n=0}^{\infty} \frac{b_n}{z^{n+1}}$ and its analytic continuation. Then $K[\Omega]$ interpolates $\{b_n\}$ relative to $\{L_n\}$ generated by $\{|W(z)|^n\}$ if and only if $B$ is analytic on $\{W(\Omega)\}^c$.

**Proof.** The sequence $\{b_n\}$ can be interpolated by $K[\Omega]$ relative to $\{f^{(n)}(0)\}$ if and only if there exists $g$ in $K[\Omega]$ such that $\{g^{(n)}(0)\} = \{b_n\}$ if and only if the function $G$, the Borel transform of $g$, defined by $G(z) = \sum_{n=0}^{\infty} \frac{b_n}{z^{n+1}}$ and its analytic continuation is analytic on $\{W(\Omega)\}^c$. Define the operator $T$ on $H(W(\Omega))$ by $[T(f)](z) = f(W(z))$. Then $T$ is interpolation preserving. Thus, $\{b_n\}$ can be interpolated by $K[\Omega]$ if and only if $B = G$ is analytic on $\{W(\Omega)\}^c$.

Note that in 1948 Buck [3] proved the necessity of Theorem 7.1 for the case $\Omega$ contains the origin and $W(0) = 0$. DeMar [10] proved the sufficiency in 1961 for the case $W(0) = 0$ and $\Omega$ is a convex set containing the origin. In 1963 DeMar [10] replaced the convexity requirement with the condition that $\Omega$ be simply connected.

The condition that $W$ be analytic and univalent on $\Omega$ in Theorem 7.1 is to assure that $K[\Omega]$ is a uniqueness class for $\{L_n\}$ generated by the kernel functions $\{(|W(z)|^n)\}$. Since we have used the concept of Polya property to expand the generating kernels for the purpose of solving the uniqueness problem, we could replace the condition with the generating kernel $m(w,z)$ having the Polya property on $\Omega_z$ as described in Section 4 (see also [6], p.33-46).

**Theorem 7.2.** Let $\Omega_w$ be a domain and $\Omega_z$ a simply connected domain in $C$. Let $m(w,z)$ be holomorphic on $\Omega_w \times \Omega_z$ and have the Polya property with
respect to $z$ on $\Omega_z$. Define the generating kernel $\{g_n(z)\}$ in (4.1) in terms of $m(w, z)$ as illustrated in Section 4 ([6, p.33 – 46]). Let $\{b_n\}$ be a sequence of complex numbers and define a function $B$ by $B(z) = \sum_{n=0}^{\infty} \frac{b_n}{z^{n+1}}$ and its analytic continuation. Then $K[\Omega_z]$ interpolates $\{b_n\}$ relative to $\{L_n\}$ generated by $\{g_n(z)\}$ if and only if $B$ is analytic on $[g_n(\Omega_z)]^c$.

**Proof.** The same as that for Theorem 7.1, using the facts stated in Section 7, especially the relationship between kernel functions having the Pólya property and operators being interpolation preserving.

**References**


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