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# CHARACTERIZATION OF MAPS HAVING THE KKM PROPERTY

BY

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**Abstract**. We characterize maps having the KKM property and then apply this characterization to construct some interesting examples showing that the KKM property is usually not preserved under operations such as product, addition and composition.

## 1. Introduction

A multimap (simply, a map) $T : X \multimap Y$ is a function from a set $X$ into the power set $2^Y$ of $Y$ having nonempty values. For a subset $A$ of $X$, $T(A)$ is defined to be $\bigcup_{x\in A} T(x)$. When $T$ is singled-valued, the notation $\multimap$ is replaced by $\rightarrow$.

The set of all nonempty finite subsets of a set $X$ is denoted by $\langle X\rangle$.

If $A$ is a subset of a vector space, then co$A$ means the convex hull of $A$. In a topological space, $\overline{A}$ denotes the closure of $A$.

The following concept of KKM property is introduced by T. H. Chang and C. L. Yen [1].

**Definition 1.1.** Let $X$ be a nonempty convex subset of a vector space and $Y$ a topological space. If $F, T : X \multimap Y$ satisfy that $T(\mathrm{co}A) \subseteq F(A)$ for any $A \in \langle X\rangle$, then $F$ is said to a generalized KKM map with respect to $T$. If the map $T : X \multimap Y$ satisfies that for any generalized KKM map $F : X \multimap Y$ with respect to $T$ the family $\{\overline{F(x)} : x \in X\}$ has the finite intersection property, then $T$ is said to have the KKM property. The class KKM$(X, Y)$ is defined to be the set of all maps $T : X \multimap Y$ having the KKM property.

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A lot of interesting and generalized results about fixed point and coincidence theory have been studied in the setting of KKM class by T. H. Chang and C. L. Yen [1]. Recently, L. J. Lin and Z. T. Yu [4] applied Proposition 7 of T. H. Chang [2] to obtain some fixed point and generalized quasi-equilibrium theorems in the KKM class. However, checking Chang's proof, we do not think his result is valid. It is the purpose of this paper to study whether the KKM class is closed under operations such as product, addition and composition. In section 2, we shall characterize maps having the KKM property. And then in section 3 we apply our characterization theorem to construct some interesting examples showing that the KKM property is usually not preserved under operations such as product, addition and composition. Specifically, counterexamples to Propositions 5 and 7 of T. H. Chang [2] will be given. Consequently, the related Theorems 6, 8, 12 and Proposition 11 of Chang's paper and the Theorems 7−13 of L. J. Lin and Z. T. Yu [4] are doubtful because their proofs are based on the aforementioned two false propositions.

In this paper, $|A|$ denotes the cardinality of a set A, and as usual, $\mathbb{Q}$ denotes the set of rational numbers and $\mathbb{R}$ the real line.

## 2. Characterization of Maps Having the KKM Property

A convex space $X$ is a nonempty convex set (in a vector space) with any topology that induces the Euclidean topology on the convex hulls of its nonempty finite subsets, (cf. M. Lassonde [3]).

Let $X$ be a nonempty convex set in a vector space and $D$ a nonempty subset of $X$. A map $F : D \multimap X$ is called a KKM map if $\mathrm{co}A \subseteq F(A)$ for any $A \in \langle D \rangle$.

The following form of KKM principle is well-known.

**Lemma 2.1.** *Let $D$ be any nonempty subset of a convex space $X$ and $F : D \multimap X$ a KKM map with closed values. Then $\{F(x) : x \in D\}$ has the finite intersection property.*

**Theorem 2.2.** *Let $X$ and $Y$ be two convex spaces and $T : X \multimap Y$ satisfy that $\overline{T(C)}$ is convex for any convex subset $C$ of $X$. Then $T \in \mathrm{KKM}(X, Y)$.*

**Proof.** Let $F : X \multimap Y$ be a generalized KKM map with respect to $T$ such that $F(x)$ is closed for each $x \in X$. We must show that $\{F(x) : x \in X\}$ has the finite intersection property. For any $A = \{x_1, \cdots, x_n\} \in \langle X \rangle$, choose $y_i \in T(x_i)$ and put $A_i = \{x_j : 1 \leq j \leq n,\, y_i \in T(x_j)\}$ for $i = 1, \cdots, n$. Define $G : Y \multimap Y$ by

$$G(y) = \begin{cases} Y, & \text{if } y \notin \{y_1, \cdots, y_n\}; \\ \bigcap_{x_j \in A_i} F(x_j), & \text{if } y = y_i,\ i = 1, \cdots, n. \end{cases}$$

It is obvious that $G(y)$ is nonempty and closed for any $y \in Y$. We claim that $G$ is a KKM map on $Y$. For this, it suffices to show that $\mathrm{co}B \subseteq G(B)$ for any $B \in \langle \{y_1, \cdots, y_n\} \rangle$. Let $B = \{y_{i_1}, \cdots, y_{i_l}\}$ be any nonempty finite subset of $\{y_1, \cdots, y_n\}$. For any $x_{j(k)} \in A_{i_k},\ k = 1, \cdots, l$, we have

$$y_{i_k} \in T(x_{j(k)}) \subseteq T(\mathrm{co}\{x_{j(1)}, \cdots, x_{j(l)}\}),$$

and so

$$\begin{aligned} \mathrm{co}\{y_{i_1}, \cdots, y_{i_l}\} &\subseteq \overline{T(\mathrm{co}\{x_{j(1)}, \cdots, x_{j(l)}\})} \\ &\subseteq \overline{\bigcup_{k=1}^{l} F(x_{j(k)})} \\ &= \bigcup_{k=1}^{l} F(x_{j(k)}), \end{aligned} \tag{2.1}$$

where the first inclusion in (2.1) is due to the hypothesis that $\overline{T(\mathrm{co}\{x_{j(1)}, \cdots, x_{j(l)}\})}$ is convex. Since $x_{j(k)} \in A_{i_k}$ is arbitrary, (2.1) implies that

$$\begin{aligned} \mathrm{co}\{y_{i_1}, \cdots, y_{i_l}\} &\subseteq \bigcup_{k=1}^{l} \Big( \bigcap_{x_{j(k)} \in A_{i_k}} F(x_{j(k)}) \Big) \\ &= \bigcup_{k=1}^{l} G(y_{i_k}), \end{aligned}$$

which shows that $G$ is a KKM map. It now follows from the KKM principle that $\{G(y) : y \in Y\}$ has the finite intersection property. In particular, $\bigcap_{i=1}^{n} G(y_i) \neq \phi$, which in turn implies that $\bigcap_{i=1}^{n} F(x_i) \neq \phi$. This shows that $T \in \mathrm{KKM}(X, Y)$.

When $X$ is an interval of $\mathbb{R}$, a better result can be derived via the order property of $\mathbb{R}$.

**Theorem 2.3.** *Let $X$ be a nonempty interval of $\mathbb{R}$ and $Y$ a topological space. If $T : X \multimap Y$ satisfies that $\overline{T([a,b])}$ is connected for any $a, b \in X$ with $a < b$, then $T \in \mathrm{KKM}(X, Y)$.*

**Proof.** Let $F : X \multimap Y$ be a generalized KKM map with respect to $T$ such that $F(x)$ is closed for any $x \in X$. We must show that $\{F(x) : x \in X\}$ has the finite intersection property. For this, it is sufficient to show that $\bigcap_{i=1}^n F(x_i) \neq \phi$ for any $n \in \mathbb{N}$ and any $x_1, \cdots, x_n$ in $X$ with $x_1 < x_2 < \cdots < x_n$. There is nothing to prove for $n = 1$. Assume $n \geq 2$. Since $[x_1, x_2] \subseteq [x_1, x_j]$ for any $j \geq 2$, we have

$$T([x_1, x_2]) \subseteq T([x_1, x_j]) \subseteq F(x_1) \cup F(x_j).$$

Thus $T([x_1, x_2]) \subseteq F(x_1) \cup (\bigcap_{j=2}^n F(x_j))$, and so

$$\overline{T([x_1, x_2])} \subseteq F(x_1) \cup (\bigcap_{j=2}^n F(x_j)),$$

once we note that $F$ is closed-valued. In case that $F(x_1) \cap (\bigcap_{j=2}^n F(x_j)) \neq \phi$, then we are done. Otherwise, since $\overline{T([x_1, x_2])}$ is connected by assumption, we have either $\overline{T([x_1, x_2])} \subseteq \bigcap_{j=2}^n F(x_j)$ or $\overline{T([x_1, x_2])} \subseteq F(x_1)$. If $\overline{T([x_1, x_2])} \subseteq \bigcap_{j=2}^n F(x_j)$, then we have $T(x_1) \subseteq \overline{T([x_1, x_2])} \subseteq \bigcap_{j=2}^n F(x_j)$, which in conjunction with the inclusion $T(x_1) \subseteq F(x_1)$ implies that

$$T(x_1) \subseteq F(x_1) \cap \Big(\bigcap_{j=2}^n F(x_j)\Big). \tag{2.2}$$

But, (2.2) contradicts the assumption that $F(x_1) \cap (\bigcap_{j=2}^n F(x_j)) = \phi$. Hence, we must have $T(x_2) \subseteq T([x_1, x_2]) \subseteq F(x_1)$, and due to $T(x_2) \subseteq F(x_2)$, it follows that

$$T(x_2) \subseteq F(x_1) \cap F(x_2). \tag{2.3}$$

Next, for any $j \geq 3$, one has

$$T([x_2, x_3]) \subseteq F(x_2) \cup F(x_j), \tag{2.4}$$

and

$$T([x_2, x_3]) \subseteq T([x_1, x_j]) \subseteq F(x_1) \cup F(x_j). \tag{2.5}$$

Taking intersection of (2.4) and (2.5), we obtain that

$$T([x_2, x_3]) \subseteq (F(x_1) \cap F(x_2)) \cup F(x_j), \quad \forall j \geq 3, \tag{2.6}$$

and therefore,

$$T([x_2, x_3]) \subseteq \overline{T([x_2, x_3])} \subseteq (F(x_1) \cap F(x_2)) \cup (\bigcap_{j=3}^{n} F(x_j)). \tag{2.7}$$

Since $\overline{T([x_2, x_3])}$ is connected and $\bigcap_{j=1}^{n} F(x_j) = \phi$, it follows from (2.7) that either $T([x_2, x_3]) \subseteq \overline{T([x_2, x_3])} \subseteq F(x_1) \cap F(x_2)$ or $T([x_2, x_3]) \subseteq \overline{T([x_2, x_3])} \subseteq \bigcap_{j=3}^{n} F(x_j)$. If it were true that $T([x_2, x_3]) \subseteq \bigcap_{j=3}^{n} F(x_j)$, then, in view of (2.3), we would have $\phi \neq T(x_2) \subseteq \bigcap_{j=1}^{n} F(x_j) = \phi$, a contradiction. Hence, we must have that $T(x_3) \subseteq T([x_2, x_3]) \subseteq F(x_1) \cap F(x_2)$, which, together with $T(x_3) \subseteq F(x_3)$, implies that

$$T(x_3) \subseteq F(x_1) \cap F(x_2) \cap F(x_3). \tag{2.8}$$

Repeating the above argument $n-1$ times, we finally obtain that $\phi \neq T(x_n) \subseteq F(x_1) \cap \cdots \cap F(x_n) = \phi$, a contradiction. This completes the proof.

Here, we like to give an example showing that the converse of Theorem 2.3 is not true. Indeed, let $T : [0, 1] \to [0, 1]$ be defined by $T(x) = \{0, 1\}$ for $x = 0$; and $T(x) = \{1\}$ for $x \in (0, 1]$. Then $T$ has the KKM property while $\overline{T([0, 1])} = \{0, 1\}$ is disconnected.

However, if we assume that $\overline{T(x)}$ is connected for any $x \in X$, then we have the following:

**Theorem 2.4.** *Let $X$ be any convex space, $Y$ a topological space and $T \in \mathrm{KKM}(X, Y)$ satisfying that $\overline{T(x)}$ is connected for any $x \in X$. Then $\overline{T(C)}$ is connected for any convex subset $C$ of $X$.*

**Proof.** On the contrary, assume that there is a convex subset $C$ of $X$ such that $\overline{T(C)}$ is disconnected. Then there exist two nonempty disjoint closed subsets $W$ and $V$ in $Y$ such that $\overline{T(C)} \subseteq W \cup V$. It is clear that we can find two points $p$ and $q$ in $C$ such that $T(p) \cap W \neq \phi$ and $T(q) \cap V \neq \phi$. Since both of $\overline{T(p)}$ and $\overline{T(q)}$ are connected by hypothesis, $T(p) \subseteq \overline{T(p)} \subseteq W$ and $T(q) \subseteq \overline{T(q)} \subseteq V$.

Define $F : X \multimap Y$ by

$$F(x) = \begin{cases} \mathrm{W}, & \text{if } x = p; \\ \mathrm{V}, & \text{if } x = q; \\ \mathrm{Y}, & \text{if } x \notin \{p, q\}. \end{cases}$$

Clearly, $F$ is closed-valued and satisfies that

$$T(\mathrm{co}A) \subseteq F(A), \quad \text{for all } A \in \langle X \rangle.$$

Thus, we have reached a contradiction to the fact that $T \in \mathrm{KKM}(X, Y)$ by noting that $F(p) \cap F(q) = \phi$. This completes the proof.

The result below is a characterization theorem for single-valued maps in the KKM class.

**Theorem 2.5.** *Let $X$ be a nonempty interval of $\mathbb{R}$, $Y$ a topological space and $T : X \to Y$. Then $T \in \mathrm{KKM}(X, Y)$ if and only if $\overline{T([a, b)])}$ is connected for any $a, b \in X$ with $a < b$.*

**Proof.** This follows immediately from Theorems 2.3 and 2.4.

## 3. Applications

Throughout this section, the following notations are used. $I$ denotes the compact interval $[0, 1]$ of $\mathbb{R}$. The relation $\sim$ on $I$ defined by $a \sim b$ if and only if $a - b \in \mathbb{Q}$ is an equivalence relation. $S$ is used to denote a complete set of representatives for equivalence classes modulo $\sim$, that is, if $P_s$ is the equivalence class associated with $s \in S$, then $\{P_s : s \in S\}$ forms a partition of $I$. We assume $0 \in S$ and define $S^*$ to be $S \setminus \{0\}$.

The following Examples 3.1 and 3.2 will show that the Cartesian product of two maps having the KKM property may fail to have the KKM property.

**Example 3.1.** Let $G = \{(x, y) \in I \times I : x = 0 \text{ or } y = 0\} \cup \{(1, 1)\}$. Since $|G| = |S|$, there is a bijection $\varphi : S \to G$. Define $h : I \to I \times I$ by $h(x) = \varphi(s)$ if $x \in P_s$. Firstly, we show that $h([a, b]) = G$ for any $a, b \in I$ with $a < b$. To see this, for any $v \in G$ choose $s \in S$ such that $\varphi(s) = v$ and then choose $x \in [a, b]$ such that $x - s \in \mathbb{Q}$. Then $x \in P_s$ and so $h(x) = \varphi(s) = v$. Next, choose two distinct points $p, q$ in $I$ such that $h(p) \in G \setminus \{(1, 1)\}, h(q) = (1, 1)$, and define

$F : I \multimap I \times I$ by

$$F(x) = \begin{cases} G \setminus \{(1,1)\}, & \text{if } x = p; \\ \{(1,1)\}, & \text{if } x = q; \\ I \times I, & \text{if } x \in I \setminus \{p, q\}. \end{cases}$$

It is easy to see that $F$ is a generalized KKM map with respect to $h$ while $F(p) \cap F(q) = \phi$. This shows that $h \notin \mathrm{KKM}(I, I \times I)$. On the other hand, let $\pi_i$ denote the projection of $\mathbb{R} \times \mathbb{R}$ onto the $i$-th coordinate for $i = 1, 2$. We have $h = (h_1, h_2)$, where $h_i = \pi_i h, i = 1, 2$. Noting that $h_i([a, b]) = \pi_i(h[a, b]) = \pi_i(G) = I$ for $a, b \in I$ with $a < b$ and $i = 1, 2$, it follows from Theorem 2.5 that $h_i \in \mathrm{KKM}(I, I)$. Therefore we have constructed two maps $h_1, h_2 \in \mathrm{KKM}(I, I)$ with $h = (h_1, h_2) \notin \mathrm{KKM}(I, I \times I)$, which shows Proposition 7 of T. H. Chang [2] is false.

**Example 3.2.** All notations are just as in Example 3.1. Define $g : I \times I \to I \times I$ by $g(x, y) = (h_1(x), h_2(y))$. Then $g \notin \mathrm{KKM}(I \times I, I \times I)$. In fact, if $g$ were in KKM $(I \times I, I \times I)$, then we would have that $g \circ \alpha \in \mathrm{KKM}(I, I \times I)$, where $\alpha : I \to I \times I$ is defined by $\alpha(x) = (x, x)$. However, $g \circ \alpha = (h_1, h_2) = h \notin \mathrm{KKM}(I, I \times I)$. Therefore, we have that $g = (h_1, h_2) \notin \mathrm{KKM}(I \times I, I \times I)$ while $h_1, h_2 \in \mathrm{KKM}(I, I)$. This shows that Proposition 5 of T. H. Chang [2] is not correct.

The example below will show that the KKM class is not closed under addition. More precisely, we will find two maps $g, h \in \mathrm{KKM}(I, I)$ but $g + h \notin \mathrm{KKM}(I, I)$.

**Example 3.3.** Since $|S^*| \geq |\mathbb{Q} \cap I|$, there is a surjection $\varphi : S^* \to \mathbb{Q} \cap I$. Define $g, h : I \to I$ respectively by

$$g(x) = \begin{cases} 0, & \text{if } x \in P_0; \\ \varphi(s), & \text{if } x \in P_s \text{ and } s \neq 0, \end{cases}$$

$$h(x) = \begin{cases} 0, & \text{if } x \in P_0; \\ 1 - \varphi(s), & \text{if } x \in P_s \text{ and } s \neq 0. \end{cases}$$

Let $a, b$ be any two points in $I$ with $a < b$. For any $q \in \mathbb{Q} \cap I$, choose $s \in S^*$ such that $\varphi(s) = q$, and then find $x \in [a, b]$ such that $x - s \in \mathbb{Q}$. So $x \in P_s$ and $g(x) = \varphi(s) = q$. Hence we have shown that $\mathbb{Q} \cap I \subseteq g([a, b]) \subseteq \mathbb{Q} \cap I$, which implies that $I = \overline{\mathbb{Q} \cap I} \subset \overline{g([a,b])} \subseteq \overline{\mathbb{Q} \cap I} = I$, that is, $\overline{g([a,b])} = I$. By Theorem 2.5, $g \in \mathrm{KKM}(I,I)$. Similarly, $h \in \mathrm{KKM}(I,I)$. On the other hand,

$$(g+h)(x) = \begin{cases} 0, & \text{if } x \in P_0; \\ 1, & \text{if } x \in P_s \text{ and } s \neq 0. \end{cases}$$

Since $\overline{(g+h)([0,1])} = \{0,1\}$ is disconnected, $g+h \notin \mathrm{KKM}(I,I)$.

We now give an example showing that the KKM class is not closed under composition.

**Example 3.4.** Choose a surjection $\varphi : S^* \to \mathbb{Q} \cap (I \setminus \{1\})$ and define $g : I \to I$ by

$$g(x) = \begin{cases} 0, & \text{if } x = 1; \\ 1, & \text{if } x \in P_0 \text{ and } x \neq 1; \\ \varphi(s), & \text{if } x \in P_s \text{ and } s \neq 0. \end{cases}$$

Just as in Example 3.3, $g \in \mathrm{KKM}(I,I)$. Now, noting that

$$g \circ g(x) = \begin{cases} 1, & \text{if } x = 1 \text{ or } x \in I \setminus \mathbb{Q}; \\ 0, & \text{if } x \in [0,1) \cap \mathbb{Q}, \end{cases}$$

we see that $\overline{g \circ g([0,1])} = \{0,1\}$ is disconnected, and hence $g \circ g \notin \mathrm{KKM}(I,I)$ by Theorem 2.5.

Here we like to mention that the map in the above Example 3.4 is fixed-point free, which serves as an example showing that the closedness assumption in Theorem 2 of T, H. Chang and C. L. Yen [1] can not be dropped.

In the remainder of this section, we are going to give an example showing that if $X$ is not an interval, then the converse of Theorem 2.4 is not true. To construct such example, we need some preliminaries.

The following Lemma 3.5 is well-known, (cf.[5, p.120]).

**Lemma 3.5.** *Let $H$ be any set with $|H| \geq 2$. Then there is a bijection $\tau : H \to H$ such that $\tau$ is fixed-point free.*

**Lemma 3.6.** *There is a bijection $\varphi : S \to I$ such that $\varphi(s) \notin P_s$ for any $s \in S$.*

**Proof.** Since $|S| = |I|$, there is a bijection $\psi : S \to I$. Let $H = \{s \in S : \psi(s) \in P_s\}$. If $H$ is empty, then $\psi$ is the desired function. Otherwise, either $|H| = 1$ or $|H| \geq 2$. In case $|H| = 1$, say, $H = \{\hat{s}\}$, choose $t \in I \setminus P_{\hat{s}}$ and find $\hat{x} \in S$ such that $\psi(\hat{x}) = t$. If $\hat{x} = \hat{s}$, then we would have $t = \psi(\hat{s})$, a contradiction. Thus $\hat{x} \neq \hat{s}$. Now define $\varphi : S \to I$ by

$$\varphi(s) = \begin{cases} t, & \text{if } s = \hat{s}; \\ \psi(\hat{s}), & \text{if } s = \hat{x}; \\ \psi(s), & \text{if } s \in S \setminus \{\hat{s}, \hat{x}\}. \end{cases}$$

Clearly, $\varphi$ is a bijection and satisfies that $\varphi(s) \notin P_s$ for each $s \in S$. If $|H| \geq 2$, then by Lemma 3.5 there is a fixed-point free bijection $\tau : H \to H$. Define $\varphi : S \to I$ by $\varphi(x) = \psi(x)$ for $x \notin H$, and $\varphi(x) = \psi(\tau(x))$ for $x \in H$. Then $\varphi$ is a bijection. Moreover, $\varphi(x) = \psi(x) \notin P_x$ for $x \notin H$, and $\varphi(x) = \psi(\tau(x)) \in P_{\tau(x)} \subseteq I \setminus P_x$ for $x \in H$. This completes the proof.

**Proposition 3.7.** *There exists a map $f : I \to I$ satisfying*

(i) $f([a, b]) = I$ *for any* $a, b \in I$ *with* $a < b$,

(ii) $f$ *is fixed-point free.*

**Proof.** By Lemma 3.6 there is a bijection $\varphi : S \to I$ such that $\varphi(s) \notin P_s$ for any $s \in S$. Define $f : I \to I$ by $f(x) = \varphi(s)$ if $x \in P_s$. For any $a, b \in I$ with $a < b$ and for any $y \in I$, choosing $s \in S$ such that $\varphi(s) = y$ and noting that $P_s \cap [a, b] \neq \phi$, we have $f(x) = y$ for any $x \in P_s \cap [a, b]$, and hence $f([a, b]) = I$. Furthermore, if $f(\hat{x}) = \hat{x}$ for some $\hat{x} \in I$, then we would have $\hat{x} = \varphi(\hat{s}) \in P_{\hat{s}}$ for some $\hat{s} \in S$, which contradicts the fact that $\varphi(s) \notin P_s$ for any $s \in S$. This completes the proof.

We are now in the position to give our final example which will show the converse of Theorem 2.4 is false.

**Example 3.8.** Let $X = I \times I, Y = I \times I$ and $A = \{(x, 1 - x) : x \in I\}, B = \{(x, 0) : x \in I\}, C = \{(0, x) : x \in I\}$ and $D = \{(x, 0) : 0 \leq x \leq \sqrt{2}\}$. For simplicity, we identify $D$ with the compact interval $[0, \sqrt{2}]$. Let h be a homeomorphism from $D$ to $I$. By Proposition 3.7, there is a map $f : I \to I$ such that $f([a, b]) = I$ for any $a, b \in I$ with $a < b$. Also, there is a continuous surjection $k : I \to A \cup B \cup C$. Then the map $\tau := k \circ f \circ h : D \to A \cup B \cup C$ satisfies that $\tau(J) = A \cup B \cup C$ for any nonempty interval $J$ of $D$. Define $g : I \times I \to I \times I$ by

$$g(x,y) = \begin{cases} (x,y), & \text{if } (x,y) \in A \cup B \cup C; \\ \tau(\sqrt{x^2+y^2}), & \text{if } (x,y) \notin A \cup B \cup C. \end{cases}$$

Let K be any nonempty convex subset of $I \times I$. If $K \subseteq A \cup B \cup C$, then $g(K) = K$. If K is not a subset of $A \cup B \cup C$, then it is quite easy to see that the image of K under the mapping $(x,y) \mapsto \sqrt{x^2+y^2}$ contains an nonempty interval of $D$, and consequently $g(K) = A \cup B \cup C$. Therefore $g$ maps any nonempty convex subset of $I \times I$ into a connected subset. Define $F : I \times I \multimap I \times I$ by

$$F(x,y) = \begin{cases} I \times I, & \text{if } (x,y) \in \\ & \ I \times I \backslash \{(0,0),(1,0),(0,1)\}; \\ \{(0,t) : 0 \le t \le 1/2\} \cup \{(t,0) : 0 \le t \le 1/2\}, & \text{if } (x,y) = (0,0); \\ \{(t,1-t) : 1/2 \le t \le 1\} \cup \{(t,0) : 1/2 \le t \le 1\}, & \text{if } (x,y) = (1,0); \\ \{(0,t) : 1/2 \le t \le 1\} \cup \{(t,1-t) : 0 \le t \le 1/2\}, & \text{if } (x,y) = (0,1). \end{cases}$$

To show that $F$ is a generalized KKM map with respect to $g$, it suffices to check that

$$g(\mathrm{co}W) \subseteq F(W), \quad \forall\ W \in \langle \{(0,0),(1,0),(0,1)\} \rangle. \tag{3.1}$$

Since (3.1) is quite clear from the definition of g and F, we omit the detail. Finally, noting $F(0,0) \cap F(1,0) \cap F(0,1) = \phi$, we conclude that $g \notin \mathrm{KKM}(I \times I, I \times I)$.

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