AN M/G/1 TYPE QUEUE WITH TIME-HOMOGENEOUS BREAKDOWNS AND DETERMINISTIC REPAIR TIMES

BY

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Abstract. We investigate the steady state behaviour of a single server queue with Poisson arrivals and arbitrary (general) service times. The system is prone to random breakdowns and just after a breakdown the server undergoes repairs of a fixed (constant) duration. The supplementary variable technique is employed to find explicitly the probability generating function of the number in the system and the mean number in the system. Some particular cases of interest are discussed and some known results are derived as special cases.

Introduction

Many authors have studied queues with server vacations or breakdowns. Gaver [6], Levy and Yechilai [10], Fuhrman [5], Doshi [4], Keilson and Servi [8], Shanthikumar [17], Cramer [3], Madan [13] and Madan and Saleh [14] are a few among several others who studied queues with server vacations. Sengupta [16] and Takine and Sengupta [18] have studied queues with service interruptions. Li et al [9] studied reliability analysis of an M/G/1 queue with breakdowns and vacations. In the present paper, we study the steady state behaviour of an M/G/1 type queue with random breakdowns and deterministic repair times of fixed length \(d(> 0)\), using the supplementary variable technique due to Cox [2]. We assume that the breakdowns are random and time-homogeneous which means that service channel may fail not only while it is working, but it may fail even when it is idle. One may encounter numerous examples in real life where the repairs are of fixed duration. For convenience, we will designate our system
as the M/G/D/1 queueing system which is briefly described by the following assumptions.

- Customers arrive at the system one by one in a Poisson stream with mean arrival rate \( \lambda (> 0) \).

- The service to customers is provided one by one on a first come, first served basis and their service times follow a general (arbitrary) distribution with distribution function \( B(v) \) and the density function \( b(v) \). Let \( \mu(x)dx \) be the conditional probability of completion of a service during the interval \((x, x + dx]\) given that the elapsed time is \( x \), so that

  \[
  \mu(x) = \frac{b(x)}{1 - B(x)},
  \]

  and, therefore,

  \[
  b(v) = \mu(v) \exp \left[ - \int_0^v \mu(x)dx \right].
  \]

- The service channel is subject to random breakdowns and the failure time distribution is exponential with mean \( 1/\alpha \). Consequently the service channel may fail any time during the time interval \((t, t + dt]\) with probability \( \alpha dt \). Further we have assumed that the breakdowns are time-homogeneous which implies that the service channel may fail any time even including the period of time when it is idle.

- We assume that whenever the service channel breaks down, it instantly undergoes a repair process and the repair times are deterministic of a constant (fixed) duration \( d (> 0) \).

- Various stochastic processes involved in the system are independent of each other.

**Definitions Concerning the States of the System**

Let \( W_n(t, x) \) be the probability that at time \( t \), there are \( n(\geq 0) \) customers in the queue excluding the one in service and the elapsed service time of this customer is \( x \). Correspondingly, \( W_n(t) = \int_0^\infty W_n(x, t)dx \) is probability that at time \( t \), there are \( n(\geq 0) \) customers in the queue excluding the one in service irrespective of the elapsed time \( x \). Let \( V_n(t) \) denote the probability that at time \( t \), there are \( n(\geq 0) \) customers in the queue and the server is under repairs. Next, let \( Q(t) \) be the probability that at time \( t \), there are no customers in the system.
and the server is idle. Finally, we suppose that $K_r$ is the probability of $r$ arrivals during a repair period of duration $d$, so that

$$K_r = \frac{e^{-\lambda d} (\lambda d)^r}{r!}, \quad r = 0, 1, 2, \ldots$$

The Steady State Equations

Assuming that the steady state exists, we let $\lim_{t \to \infty} W_n(t, x) = W_n(x)$, $\lim_{t \to \infty} W_n(t) = W_n$, $\lim_{t \to \infty} V_n(t) = V_n$ and $\lim_{t \to \infty} Q(t) = Q$, so that $W_n(x)$, $W_n$, $V_n$ and $Q$ now denote the corresponding steady state probabilities. Then following Cox [2], we obtain the following set of steady state equations:

$$\frac{d}{dx}W_n(x) + (\lambda + \alpha + \mu(x))W_n(x) = \lambda W_{n-1}(x), \quad n \geq 1, \quad (3)$$

$$\frac{d}{dx}W_0(x) + (\lambda + \alpha + \mu(x))W_0(x) = 0, \quad (4)$$

$$(\lambda + \alpha)Q = \int_0^\infty W_0(x)\mu(x)dx + V_0K_0, \quad (5)$$

$$V_n = \alpha W_{n-1}, \quad n \geq 1, \quad (6)$$

$$V_0 = \alpha Q. \quad (7)$$

The above equations are to be solved subject to the following boundary conditions:

$$W_n(0) = \int_0^\infty W_{n+1}(x)\mu(x)dx + V_0K_{n+1} + V_1K_n + \cdots + V_nK_1 + V_{n+1}K_0, \quad n \geq 1, \quad (8)$$

$$W_0(0) = \int_0^\infty W_1(x)\mu(x)dx + V_0K_1 + V_1K_0 + \lambda Q. \quad (9)$$

The Steady State Probability Generating Functions

We define the following probability generating functions:

$$W(x, z) = \sum_{n=0}^\infty W_n(x, z)z^n, \quad W(z) = \sum_{n=0}^\infty W_n z^n, \quad (10a)$$

$$V(z) = \sum_{n=0}^\infty V_n z^n, \quad |z| \leq 1. \quad (10b)$$

On multiplying equations (3) and (4) by $z^n$, summing over $n$ and using (10a), we have

$$\frac{d}{dx}W(x, z) + (\lambda - \lambda z + \alpha + \mu(x))W(x, z) = 0. \quad (11)$$
Similarly, multiplying equation (6) by $z^n$, summing over $n$ and using (10a) and (10b), we get

$$V(z) = \alpha z W(z) + \alpha Q.$$  \hfill (12)

In the same manner as above, equations (8) and (9), on using (10a), yield

$$z W(0, z) = \int_0^\infty W(x, z) \mu(x) dx - \int_0^\infty W_0(x) \mu(x) dx + V(z) e^{-\lambda d(1-z)} - V_0 K_0 + \lambda z Q,$$  \hfill (13)

wherein we have used the fact that

$$\sum_0^\infty K_n z^n = \sum_0^\infty e^{-\lambda d \frac{(\lambda d)n}{n!}} z^n = e^{-\lambda d(1-z)}.$$

Using equation (5) we can re-write equation (13) as

$$z W(0, z) = \int_0^\infty W(x, z) \mu(x) dx + V(z) e^{-\lambda d(1-z)} + [\lambda(z - 1) - \alpha] Q.$$  \hfill (14)

Now, we integrate equation (11) between the limits 0 and $x$, and obtain

$$W(x, z) = W(0, z) e^{-(\lambda - \lambda z + \alpha) - \int_0^x \mu(t) dt},$$  \hfill (15)

where $W(0, z)$ is given by (14).

Then, we multiply both sides of equation (15) by $\mu(x)$ and integrate with respect to $x$. Then by virtue of equations (2) and (15), we have

$$\int_0^\infty W(x, z) \mu(x) dx = \int_0^\infty W(0, z) e^{-(\lambda - \lambda z + \alpha) - \int_0^x \mu(t) dt} \mu(x) dx$$

$$= W(0, z) \int_0^\infty e^{-(\lambda - \lambda z + \alpha)} \mu(x) e^{-\int_0^x \mu(t) dt} dx.$$  \hfill (16a)

Now by virtue of equation (2), we have $\mu(x) e^{-\int_0^x \mu(t) dt} = b(x)$ and therefore, (16a) yields

$$\int_0^\infty W(x, z) \mu(x) dx = W(0, z) \int_0^\infty e^{-\lambda - \lambda z + \alpha} b(x) dx$$

$$= W(0, z) B(\lambda - \lambda z + \alpha),$$  \hfill (16b)

where $B(\lambda - \lambda z + \alpha) = E(e^{-(\lambda - \lambda z + \alpha)x}) = \int_0^\infty e^{-(\lambda - \lambda z + \alpha)x} b(x) dx$ is the Laplace transform of the service time density function $b(x)$.  


Then using (16b) in equation (14) and simplifying, we have

$$W(0, z) = \frac{V(z)e^{-\lambda(1-z)} - (\lambda + \alpha - \lambda z)Q}{z - \overline{b}(\lambda + \alpha - \lambda z)}. \quad (17)$$

Next, we integrate equation (15) with respect to $x$ by parts and get

$$W(z) = W(0, z) \left[ \frac{1 - \overline{b}(\lambda + \alpha - \lambda z)}{\lambda + \alpha - \lambda z} \right]. \quad (18)$$

Using equation (17), equation (18) becomes

$$W(z) = \left[ \frac{1 - \overline{b}(\lambda + \alpha - \lambda z)}{\lambda + \alpha - \lambda z} \right] \left[ \frac{V(z)e^{-\lambda(1-z)} - (\lambda + \alpha - \lambda z)Q}{z - \overline{b}(\lambda + \alpha - \lambda z)} \right]. \quad (19)$$

Then substituting for $V(z)$ from (12) into (19), we have

$$W(z) = \left[ \frac{1 - \overline{b}(\lambda + \alpha - \lambda z)}{\lambda + \alpha - \lambda z} \right] \left[ \frac{(\alpha zW(z) + \alpha Q)e^{-\lambda(1-z)} - (\lambda + \alpha - \lambda z)Q}{z - \overline{b}(\lambda + \alpha - \lambda z)} \right]. \quad (20)$$

Equation (20) is further simplified as

$$W(z) = \left[ \frac{1 - \overline{b}(\lambda + \alpha - \lambda z)}{\lambda + \alpha - \lambda z} \right] \left[ \frac{(\alpha zW(z) + \alpha Q)e^{-\lambda(1-z)} - (\lambda + \alpha - \lambda z)Q}{\lambda + \alpha - \lambda z} \right]. \quad (21)$$

In order to determine the only unknown constant $Q$, we shall use the normalizing condition

$$W(1) + V(1) + Q = 1. \quad (22)$$

For that purpose, from (21), we find

$$W(1) = \lim_{z \to 1} W(z) = \frac{(1 - \overline{b}(\alpha))(\lambda + \lambda ad)Q}{\alpha \overline{b}(\alpha) - (1 - \overline{b}(\alpha))(\lambda + \lambda ad)}, \quad (23)$$

where $\overline{b}(\alpha) = E(e^{-\alpha x}) = \int_0^\infty e^{-\alpha x} b(x) dx.$

Then using (23) in equation (12), we obtain on simplifying

$$V(1) = \lim_{z \to 1} V(z) = \alpha W(1) + \alpha Q = \frac{\alpha \overline{b}(\alpha)Q}{\alpha \overline{b}(\alpha) - (1 - \overline{b}(\alpha))(\lambda + \lambda ad)}. \quad (24)$$

Finally, using (23) and (24), equation (22) yields on simplifying

$$Q = \frac{\alpha \overline{b}(\alpha) - (1 - \overline{b}(\alpha))(\lambda + \lambda ad)}{\alpha(\alpha + 1)\overline{b}(\alpha)} \quad (25)$$
which is the steady state probability that the server is idle but operative. Then we substitute the value of \( Q \) from (25) into (21) and have

\[
W(z) = \frac{[1 - \bar{b}(\lambda + \alpha - \lambda z)][ae^{-\lambda d(1-z)} + (\lambda + \alpha - \lambda z)]}{\lambda z \alpha e^{-\lambda d(1-z)}}. \quad (26)
\]

Next, using equation (26) in (12) and simplifying, we have

\[
V(z) = \frac{\alpha(\lambda + \alpha - \lambda z)\bar{b}(\lambda - \lambda z + \alpha)(z-1)}{\lambda z \alpha e^{-\lambda d(1-z)}}. \quad (27)
\]

Thus \( W(z) \) and \( V(z) \) have been completely and explicitly determined in (26) and (27).

Let \( P_q(z) = W(z) + V(z) \) denote the probability generating function of the queue length irrespective of whether the server is operative or in the failed state. Then, adding (26) and (27) and simplifying, we have

\[
P_q(z) = \frac{[1 - \bar{b}(\lambda + \alpha - \lambda z)][ae^{-\lambda d(1-z)} - (\lambda + \alpha - \lambda z)] + \alpha(\lambda + \alpha - \lambda z)\bar{b}(\lambda + \alpha - \lambda z)(z-1)}{\lambda + \alpha - \lambda z}[z - \bar{b}(\lambda + \alpha - \lambda z) - [1 - \bar{b}(\lambda + \alpha - \lambda z)]\alpha e^{-\lambda d(1-z)}} \times \frac{\alpha \bar{b}(\alpha) - (1 - \bar{b}(\alpha))(\lambda + \alpha d)}{\alpha(\alpha + 1)\bar{b}(\alpha)}. \quad (28)
\]

Further, let \( P(z) \) denote the probability generating function of the number in the system. Then from equations (25) and (28) and simplifying we have

\[
P(z) = Q + zP_q(z)
= \frac{(\lambda + \alpha - \lambda z)\bar{b}(\lambda + \alpha - \lambda z)(z-1)(1 + \alpha z)}{\lambda + \alpha - \lambda z}[z - \bar{b}(\lambda + \alpha - \lambda z + \alpha) - [1 - \bar{b}(\lambda - \lambda z + \alpha)]\alpha e^{-\lambda d(1-z)}} \times \frac{\alpha \bar{b}(\alpha) - (1 - \bar{b}(\alpha))(\lambda + \alpha d)}{\alpha(\alpha + 1)\bar{b}(\alpha)}. \quad (29)
\]

Special Cases

Case I. Exponential service, random breakdowns and deterministic repair times.

In the case of exponential service, we have \( \bar{b}(\alpha) = \int_{0}^{\infty} e^{-\alpha x} \mu e^{-\mu x} dx = \frac{\mu}{\alpha + \mu} \) and similarly \( \bar{b}(\lambda + \alpha - \lambda z) = \frac{\mu}{\lambda + \alpha + \mu - \lambda z} \). With these substitutions into the results (25) to (29), we obtain

\[
Q = \frac{\alpha \left( \frac{\mu}{\alpha + \mu} \right) - \left( 1 - \left[ \frac{\mu}{\alpha + \mu} \right] \right) (\lambda + \alpha d)}{\alpha(\alpha + 1) \left( \frac{\mu}{\alpha + \mu} \right)}, \quad (30)
\]
which further simplifies to
\[
Q = \frac{\mu - (\lambda + \lambda \alpha d)}{(\alpha + 1)\mu},
\]
and
\[
W(z) = \frac{\lambda + \alpha - \lambda z}{\lambda + \alpha - \lambda z} \left( \frac{\alpha e^{-\lambda d(1-z)}}{\lambda + \alpha - \lambda z} + (\lambda + \alpha - \lambda z) \right) \left( \frac{\mu - (\lambda + \lambda \alpha d)}{(\alpha + 1)\mu} \right) (\alpha + 1)\mu, \tag{31}
\]
\[
V(z) = \frac{\lambda + \alpha - \lambda z}{\lambda + \alpha - \lambda z} \left( \frac{\alpha e^{-\lambda d(1-z)}}{\lambda + \alpha - \lambda z} + (\lambda + \alpha - \lambda z) \right) \left( \frac{\mu - (\lambda + \lambda \alpha d)}{(\alpha + 1)\mu} \right) (\alpha + 1)\mu, \tag{32}
\]
\[
P_q(z) = \frac{\lambda + \alpha - \lambda z}{\lambda + \alpha - \lambda z} \left( \frac{\alpha e^{-\lambda d(1-z)}}{\lambda + \alpha - \lambda z} + (\lambda + \alpha - \lambda z) \right) \left( \frac{\mu - (\lambda + \lambda \alpha d)}{(\alpha + 1)\mu} \right) (\alpha + 1)\mu, \tag{33}
\]
\[
P(z) = \frac{\lambda + \alpha - \lambda z}{\lambda + \alpha - \lambda z} \left( \frac{\alpha e^{-\lambda d(1-z)}}{\lambda + \alpha - \lambda z} + (\lambda + \alpha - \lambda z) \right) \left( \frac{\mu - (\lambda + \lambda \alpha d)}{(\alpha + 1)\mu} \right) (\alpha + 1)\mu, \tag{34}
\]
\[
\lambda + \alpha - \lambda z \left( \frac{\alpha e^{-\lambda d(1-z)}}{\lambda + \alpha - \lambda z} + (\lambda + \alpha - \lambda z) \right) \left( \frac{\mu - (\lambda + \lambda \alpha d)}{(\alpha + 1)\mu} \right) (\alpha + 1)\mu, \tag{35}
\]
\[
\lambda + \alpha - \lambda z \left( \frac{\alpha e^{-\lambda d(1-z)}}{\lambda + \alpha - \lambda z} + (\lambda + \alpha - \lambda z) \right) \left( \frac{\mu - (\lambda + \lambda \alpha d)}{(\alpha + 1)\mu} \right) (\alpha + 1)\mu, \tag{36}
\]
\[
\lambda + \alpha - \lambda z \left( \frac{\alpha e^{-\lambda d(1-z)}}{\lambda + \alpha - \lambda z} + (\lambda + \alpha - \lambda z) \right) \left( \frac{\mu - (\lambda + \lambda \alpha d)}{(\alpha + 1)\mu} \right) (\alpha + 1)\mu, \tag{37}
\]
\[
\lambda + \alpha - \lambda z \left( \frac{\alpha e^{-\lambda d(1-z)}}{\lambda + \alpha - \lambda z} + (\lambda + \alpha - \lambda z) \right) \left( \frac{\mu - (\lambda + \lambda \alpha d)}{(\alpha + 1)\mu} \right) (\alpha + 1)\mu, \tag{38}
\]

Case II. M/G/1 Queue with no breakdowns.

If the system suffers no breakdowns, then letting \( \alpha = 0 \) in the main results (25) to (29), we obtain
\[
V(z) = 0, \tag{39}
\]

\[
Q = \frac{\mu - \lambda}{\mu}, \tag{40}
\]

\[
P_q(z) = W(z) = \frac{[1 - b(\lambda - \lambda z)][1 - \lambda/\mu]}{[z - b(\lambda - \lambda z)]}, \tag{41}
\]

\[
P(z) = \frac{(1 - z)b(\lambda - \lambda z)(1 - \lambda/\mu)}{b(\lambda - \lambda z) - z}. \tag{42}
\]

The result in equation (41) is a known result for the M/G/1 queue, ([see Kashyap and Chaudhry [7], equation (26), p.58]). And the result in equation (42) is the well-known Pollaczek-Khinchine formula, ([see Medhi [15], equation 6.7, p.313 or Choi and Park [1], p.225]).

It may be noted that corresponding results for many particular cases of interest such as M/Ek/D/1, M/M/D/1 and M/D/D/1, etc. can be derived from the main results found in equations (27) and (29) by substituting the appropriate values of \( b(\lambda - \lambda z) \).
Acknowledgment

The author wishes to convey his sincere thanks and gratitude to the Managing Editor and the learned referee for his valuable comments and suggestions to revise the paper in the present form.

References


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