P-CONNECTEDNESS IN \(L\)-TOPOLOGICAL SPACES

BY

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Abstract. In this paper, the P-connectedness of \(L\)-subsets in \(L\)-topological spaces is introduced and studied. It preserves some fundamental properties of connected sets in general topological spaces. Especially, the famous K. Fan’s Theorem holds for P-connectivity.

1. Introduction

Connectivity is one of the important notions in topology. Many workers have presented various kinds of connectivities ([1, 2, 7, 9, 10]) in \(L\)-topological spaces in the Chang’s [7] sense. But it is necessary to point out that the connectivities above were defined only for the case of \(L = [0, 1]\). And we must also say that the famous K. Fan’s Theorem has not been given yet. We know in general topology, there are different ways to describe connectivity of a subset. Maybe, K.Fan’s Theorem is the most interesting one, which has clear geometrical characterization. In general \(L\)-topological spaces, where \(L\) is a fuzzy lattice, Wang introduced connectivity in [12]. Later on, we introduced strongly connectivity and I type of strongly connectivity in [4, 5]. The main purpose of this paper is to introduce the P-connectivity in \(L\)-topological spaces. It preserves some fundamental properties of connected sets in general topological spaces. Especially, the famous K.Fan’s Theorem holds for P-connectivity. Also we discuss relations between P-connected set and strongly connected set, I type of strongly connected set.
2. Preliminaries

In this paper, \( L \) will denote a fuzzy lattice, i.e., completely distributive lattice with order-reversing involutions “\( \prime \)”. 0 and 1 denote the smallest element and the largest element in \( L \), respectively. Let \( X \) be a nonempty crisp set, \( L^X \) the set of all \( L \)-subsets on \( X \) and \( M(L) \) the set of all nonzero irreducible elements of \( L \). Put

\[
M^*(L^X) = \{ x_\alpha : x \in X \text{ and } \alpha \in M(L) \}.
\]

Let \((L^X, \delta)\) be an \( L \)-topological space, briefly \( L \)-ts. \( A, A^-, A^- \) and \( A' \) will denote the interior, closure, semiclosure and complement of the \( L \)-subset \( A \), respectively. \( A \in L^X \) is called a preopen (preclosed) set iff \( A \leq A^- \) (\( A \geq A^- \)). The pre-closure of the \( L \)-subset \( A \) is the intersection of all preclosed sets, each containing \( A \) ([4]). It will be denoted by \( A^\sim \).

Definition 2.1. (Wang [11]) Let \( L_1 \), and \( L_2 \) be fuzzy lattices. A mapping \( f : L_1 \rightarrow L_2 \) is called an order-homomorphism if the following conditions hold:

1. \( f(0) = 0 \).
2. \( f(\bigsqcup A_i) = \bigsqcup f(A_i) \) for \( \{A_i\} \subset L_1 \).
3. \( f^{-1}(B') = (f^{-1}(B))' \) for each \( B \in L_2 \).

Definition 2.2. (Bai [3]) Let \((L^X_1, \delta)\) and \((L^Y_2, \tau)\) be two \( L \)-ts’s and \( f : (L^X_1, \delta) \rightarrow (L^Y_2, \tau) \) an order-homomorphism. \( f \) is called a P-irresolute order-homomorphism if \( f^{-1}(B) \) is a preopen set of \( L^X_1 \) for each preopen set \( B \) of \( L^Y_2 \).

Theorem 2.3. (Bai [3]) An order-homomorphism \( f : (L^X_1, \delta) \rightarrow (L^Y_2, \tau) \) is P-irresolute iff \( f^{-1}(B') \leq f^{-1}(B^\sim) \), for each \( B \in L^Y_2 \).

Definition 2.4. (Bai [3]) Let \((L^X, \delta)\) be an \( L \)-ts, \( x_\lambda \in M^*(L^X) \) and \( P \) a preclosed set in \( L^X \). \( P \) is called a preclosed remoted-neighborhood, or briefly, \( PRN \) of \( x_\lambda \), if \( x_\lambda \notin P \). The set of all \( PRN \)’s of \( x_\lambda \) will be denoted by \( \zeta(x_\lambda) \).

3. P-Connectedness of \( L \)-Subsets

Definition 3.1. Let \((L^X, \delta)\) be an \( L \)-ts and \( A, B \in L^X \). Then \( A \) and \( B \) are said to be P-separated if \( A^\sim \wedge B = A \wedge B^\sim = 0 \).
Corollary 3.2. Let \((L^X, \delta)\) be an \(L\)-ts and \(A, B \in L^X\). If \(A\) and \(B\) are separated and \(C \leq A, D \leq B\), then \(C\) and \(D\) are also separated.

Definition 3.3. Let \((L^X, \delta)\) be an \(L\)-ts and \(A \in L^X\). \(A\) is called a P-connected set if \(A\) cannot be represented as a union of two P-separated non-null sets. Specifically, when \(A = 1\) is P-connected, we call \((L^X, \delta)\) a P-connected space.

Theorem 3.4. Let \((L^X, \delta)\) be an \(L\)-ts. Then the following are equivalent:
(1) \((L^X, \delta)\) is not a P-connected space.
(2) There exist two non-null preclosed sets \(A\) and \(B\) such that \(A \lor B = 1\) and \(A \land B = 0\).
(3) There exist two non-null preopen sets \(A\) and \(B\) such that \(A \lor B = 1\) and \(A \land B = 0\).

Proof. This is analogous to the proof of Theorem 2.5 in [5].

Theorem 3.5. Let \(A\) be a non-null P-connected set in \(L\)-ts \((L^X, \delta)\), and \(C\) and \(D\) be two P-separated sets in \((L^X, \delta)\). If \(A \leq C \lor D\), then \(A \leq C\) or \(A \leq D\).

Proof. This is easy.

Theorem 3.6. Let \(A\) be a P-connected set in an \(L\)-ts \((L^X, \delta)\). If \(A \leq B \leq A^\circ\), then \(B\) is also P-connected in \((L^X, \delta)\).

Proof. This is analogous to the proof of Theorem 6.1.7 in [8].

Theorem 3.7. Let \(\{A_t : t \in T\}\) be a family of P-connected sets in \(L\)-ts \((L^X, \delta)\). Suppose there is an \(s \in T\) such that \(A_t\) and \(A_s\) are not P-separated for each \(t \neq s\), then \(\bigvee_{t \in T} A_t\) is P-connected.

Proof. This is analogous to the proof of Theorem 6.1.8 in [8].

The following corollary is obvious.

Corollary 3.8. Let \(\{A_t : t \in T\}\) be a family of P-connected sets in \(L\)-ts \((L^X, \delta)\). If \(\bigcap_{t \in T} A_t \neq 0\), then \(\bigvee_{t \in T} A_t\) is P-connected.
Theorem 3.9. Let \( f : L_X^1 \to L_Y^2 \) be a P-irresolute order-homomorphism. If \( A \) is P-connected in \( L_X^1 \), then \( f(A) \) is P-connected in \( L_Y^2 \).

**Proof.** This is analogous to the proof of Theorem 6.1.18 in [8].

Corollary 3.10. Let \( f : L_X^1 \to L_Y^2 \) be a P-irresolute order-homomorphism and onto. If \( L_X^1 \) is a P-connected space, then so is \( L_Y^2 \).

**Proof.** This is immediate from Theorem 3.9.

Now, K. Fan’s Theorem will be extended to the P-connectedness of \( L \)-subsets in \( L \)-ts. \( M^*(A) \) denotes the set of all points of \( A \), \( \zeta(x) \) denotes the set of all PCRNs of \( x \) for each \( x \in M^*(A) \).

Theorem 3.11. Let \((L^X, \delta)\) be an \( L \)-ts and \( A \in L^X \). Then \( A \) is P-connected iff for each pair \( a, b \) of points of \( M^*(A) \) and each mapping

\[ P : M^*(A) \to \bigcup \{ \zeta(x) : x \in M^*(A) \}, \]

where \( P(x) \in \zeta(x) \) for each \( x \in M^*(A) \), there exists in \( M^*(A) \) a finite number of points \( x_1 = a, x_2, \ldots, x_n = b \) such that

\[ A \nsubseteq P(x_i) \lor P(x_{i+1}), \quad i = 1, 2, \ldots, n - 1. \] (i)

**Proof.** Sufficiency. Suppose that \( A \) is not P-connected. Then there are \( B, C \in L^X \) and \( B \neq 0, C \neq 0 \) such that

\[ B^{-} \land C = B \land C^{-} = 0 \quad \text{and} \quad A = B \lor C. \]

Consider the mapping

\[ P : M^*(A) \to \bigcup \{ \zeta(x) : x \in M^*(A) \}, \]

defined by

\[ P(x) = \begin{cases} C^{-}, & \text{if } x \leq B, \\ B^{-}, & \text{if } x \leq C. \end{cases} \]

By \( B^{-} \land C = B \land C^{-} = 0 \), we have \( x \nsubseteq P(x) \). Since \( P(x) \) is a preclosed set, \( P(x) \in \zeta(x) \) for each \( x \in M^*(A) \). Take the point \( a \) out of \( B \) and take
the point \( b \) out of \( C \). Then \( a, b \in M^*(A) \). Since for arbitrary finite points \( x_1 = a, x_2, \ldots, x_n = b \), either \( x_i \leq B \) or \( x_i \leq C \) (\( i = 1, \ldots, n \)) must be held, \( P(x_i) = C \) or \( P(x_i) = B \). But \( P(x_1) = C \) and \( P(x_n) = B \), hence there exists \( 1 \leq j \leq n - 1 \) such that \( P(x_j) = C \) and \( P(x_{j+1}) = B \). This shows that

\[
A = B \lor C \leq P(x_j) \lor P(x_{j+1}),
\]

a contradiction. Thus sufficiency is proved.

Necessity. Suppose that condition of theorem is not held, i.e., there are points \( a, b \in M^*(A) \), \( a \neq b \) and there is a mapping

\[
P : M^*(A) \to \bigcup \{ \zeta(x) : x \in M^*(A) \},
\]

where \( P(x) \in \zeta(x) \) for each \( x \in M^*(A) \), such that (i) is not held for arbitrary finite points \( x_1, \ldots, x_n \in M^*(A) \). For the sake of convenience, we follow the agreement that for arbitrary \( a, b \in M^*(A) \), \( a \) and \( b \) are joined if there are finite points \( x_1, \ldots, x_n \in M^*(A) \) such that (i) holds. Otherwise, \( a \) and \( b \) are not joined. Let

\[
\mu = \{ x \in M^*(A) : a \text{ and } x \text{ are joined} \},
\]

\[
\nu = \{ x \in M^*(A) : a \text{ and } x \text{ are not joined} \},
\]

\[
B = \lor \mu,
\]

\[
C = \lor \nu.
\]

Obviously, \( a \) and \( a \) are joined and so \( a \in \mu \) and \( a \leq B \). By hypothesis \( a \) and \( b \) are not joined, and so \( b \in \nu \) and \( b \leq C \). Hence \( B \neq 0, C \neq 0 \). Since for each \( x \in M^*(A) \) or \( x \in \mu \), or \( x \in \nu \), \( A = B \lor C \). Now we need only prove

\[
B \land C = B \land C = 0.
\]

Suppose that \( B \land C \neq 0 \), and for each \( x \leq B \land C \). By \( x \leq B \land C \), we have \( B \not\subseteq P(x) \), and so there is \( y \in \mu \) such that \( y \not\subseteq P(x) \). Hence \( y \not\subseteq P(x) \lor P(y) \) and \( y \leq B \leq A \). Thus, \( A \not\subseteq P(x) \lor P(y) \). \( y \) and \( a \) are joined so \( a \) and \( x \) are joined.

On the other hand, by \( x \leq C \), we have \( C \not\subseteq P(x) \), and so there is \( z \in \nu \) such that \( z \not\subseteq P(x) \). Hence, \( z \not\subseteq P(x) \lor P(z) \) and \( z \leq C \leq A \). Thus, \( A \not\subseteq P(x) \lor P(z) \). By \( x \) and \( a \) are joined, \( a \) and \( z \) are joined. This contradicts the \( z \in \nu \). Thus,
$B \leadsto \land C = 0$. In a similar way we can prove the $B \land C^{-} = 0$. Thus necessity is proved.

4. Mutual Relations

From [4, 5] we have known that every I type of strongly connected set is strongly connected and every strongly connected set is connected in $L$-topological spaces. Now we discuss relations between P-connected set and strongly connected set, I type of strongly connected set.

Remark 4.1. (1) Clearly, every P-connected set is strongly connected ([4]) in $L$-ts. That the converse need not be true.
(2) P-connected set and I type of strongly connected sets ([5]) in $L$-ts are independent notions.

Example 4.2. Let $X = \{x, y\}$ and $L = \{0, a, b, 1\}$, and define $0 < a < 1$, $0 < b < 1, 0' = 1, 1' = 0, a' = a$ and $b' = b$. Put $A, B, C, D \in L^X$ defined as follows:

\begin{align*}
A(x) &= a, \quad A(y) = b; \\
B(x) &= 1, \quad B(y) = 0; \\
C(x) &= a, \quad C(y) = 0; \\
D(x) &= 0, \quad D(y) = b.
\end{align*}

Then $\delta = \{0, B, 1\}$ is a topology on $L^X$. We can easily show that $A$ is I type of strongly connected sets in $L$-ts $(L^X, \delta)$. In fact, $A$ can only be expressed as the union of two disjoint non-null $L$-subsets $C$ and $D$, i.e.,

\begin{align*}
A = C \lor D, \quad C \land D = 0, \quad C \neq 0, \quad D \neq 0.
\end{align*}

Simple computations give $C_\land = 1$, and so $C_\land \land D \neq 0$, i.e., $C$ and $D$ are not I type of weakly separated. Hence $A$ is I type of strongly connected. And so $A$ is strongly connected. Clearly, $C$ and $D$ are preclosed sets, i.e., $C^{-} = C$ and $D^{-} = D$. Hence,

\begin{align*}
C^{-} \land D = C \land D^{-} = 0,
\end{align*}
i.e., $C$ and $D$ are P-separated. Thus, $A$ is not P-connected.

**Example 4.3.** Let $X = \{x, y\}, L = \{0, 1/7, 2/7, 3/7, 4/7, 5/7, 6/7, 1\}, \forall a \in L, a' = 1 - a$ and $A, C, D \in L^X$ defined as follows:

$$A(x) = 4/7, \quad A(y) = 3/7;$$
$$C(x) = 3/7, \quad C(y) = 0;$$
$$D(x) = 6/7, \quad D(y) = 0.$$

Then $\delta = \{0, C, D, 1\}$ is a topology on $L^X$. We show that $A$ is a P-connected set in $L$-ts $(L^X, \delta)$. In fact, $A$ can only be expressed as the union of two disjoint non-null $L$-subsets, i.e.,

$$A = P \lor Q, \quad P \land Q = 0, \quad P \neq 0, \quad Q \neq 0,$$

where $P(x) = 4/7, P(y) = 0; Q(x) = 0, Q(y) = 3/7$. By easy computations it follows that $P^\sim = C'$. Then $P^\sim \land Q \neq 0$, i.e., $P$ and $Q$ are not P-separated. Hence $A$ is P-connected. Again, $C'$ and $D'$ are closed sets, and

$$C = (C')^0 \leq P \leq C',$$
$$0 = (D')^0 \leq Q \leq D'.$$

And so $P$ and $Q$ are semiclosed sets, i.e., $P_\sim = P$ and $Q_\sim = Q$. Hence,

$$P_\sim \land Q = P \land Q_\sim = 0,$$

i.e., $P$ and $Q$ are I type of weakly separated. Thus $A$ is not a I type of strongly connected set in $L$-ts $(L^X, \delta)$.

**References**


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