FACTORIZATION OF DIFFERENTIAL OPERATORS FOR SOME SPECIAL FUNCTIONS

BY

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Abstract. Let \( \{S_n(x)\}_{n=0}^{\infty} \) be a sequence of special functions depending on a parameter \( n \), and suppose that there exist two sequences of differential operators \( E_n^- \) and \( E_n^+ \) satisfying the following properties:

\[
E_n^-(S_n(x)) = S_{n-1}(x),
\]

\[
E_n^+(S_n(x)) = S_{n+1}(x).
\]

By constructing these two operators, and using the factorization method ([10]), we determine the differential equations satisfied by some special functions, including some new classes recently introduced by G. Dattoli et al. by using the so called Monomiality principle [4]-[6].

1. Introductory Definitions

Let \( \{S_n(x)\}_{n=0}^{\infty} \) be a sequence of special functions depending on a parameter \( n \). For \( n = 0, 1, 2, \ldots \), we consider two sequences of operators \( E_n^- \) and \( E_n^+ \) satisfying the following properties:

\[
E_n^-(S_n(x)) = S_{n-1}(x),
\]

\[
E_n^+(S_n(x)) = S_{n+1}(x).
\]

\( E_n^- \) and \( E_n^+ \) play the role analogous to that of derivative and multiplicative operators respectively on monomials. In recent papers the monomiality principle and

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the associated operational rules were used to explore new classes of isospectral problems leading to non trivial generalizations of special functions (see e.g. [1]-[5]-[2]-[3]). Most of the properties of functions associated with $E_n^-$ and $E_n^+$ can be deduced by using operational rules with these two operators. The operators defined in this paper are varying with the above mentioned parameter $n$.

The iterations of $E_n^-$ and $E_n^+$ to $S_n(x)$ give the following relations:

\[
(E_{n+1}^- E_n^+) S_n(x) = S_n(x),
\]

\[
(E_{n-1}^- E_n^-) S_n(x) = S_n(x),
\]

\[
(E_{n+1}^- E_{n+1}^- E_{n-1}^- E_n^-) S_n(x) = S_{n-1}(x),
\]

\[
(E_{n-1}^+ E_{n-2}^+ E_{n-1}^- E_0^+) S_0(x) = S_n(x).
\]

These operational relations will be used to derive differential equations, in general of higher order, satisfied by some special functions. The classical factorization method introduced in [10] was applied to construct second-order differential equations.

2. Operators $E_n^-$ and $E_n^+$

In the following we will use special functions satisfying a $(d+2)$-term recurrence relation with $d \geq 1$. The number of the terms of recurrence is fixed and greater than or equal to 3. There are many classes of special functions with this type of recursion formulas such as orthogonal polynomials and confluent hypergeometric functions (see e.g. [9]).

The following main result, proved in [9], will be used:

**Theorem 2.1.** Let \( \{S_n(x)\}_{n \geq 0} \) be a sequence of functions satisfying the following $(d+2)$-term recurrence relation

\[
S_{n+d+1}(x) = (x - \alpha_{n+d})S_{n+d}(x) + \sum_{k=1}^{d} \beta_{n+d-k} S_{n+d-k}(x),
\]

with some proper initial conditions. If there is a differential operator $E_n^-$ such that

\[
E_n^- S_n(x) = S_{n-1}(x),
\]
then the functions of the sequence \( \{S_n(x)\}_{n \geq 0} \) satisfy the following differential equation

\[
(E_{n+1}^- E_n^+) S_n(x) = S_n(x), \quad (n \geq d), \tag{2.3}
\]

where

\[
E_{n+d}^+ = (x - \alpha_{n+d}) + \sum_{k=1}^{d} \beta_{n+d-k} \prod_{j=1}^{k} E_{n+d-k+j}^- . \tag{2.4}
\]

If there is an operator \( E_n^+ \) such that

\[
E_n^+ S_n(x) = S_{n+1}(x), \tag{2.5}
\]

then the operator \( E_{n+1}^- , \forall n \geq 0, \) is given by

\[
E_{n+1}^- = \frac{1}{\beta_n} \left\{ \prod_{k=1}^{d} E_{n+k}^+ - \left[ (x - \alpha_{n+d}) \prod_{k=1}^{d-1} E_{n+k}^+ + \sum_{k=1}^{d-1} \beta_{n+d-k} \prod_{j=1}^{d-k-1} E_{n+j}^+ \right] \right\} . \tag{2.6}
\]

The authors, in their paper [9], apply the above stated result to the classical orthogonal polynomials, \( d \)-orthogonal polynomials, confluent hypergeometric functions and hypergeometric functions.

In the following section we illustrate the factorization method for some other classes of special functions. In each case, the operators \( E_n^- \) and \( E_n^+ \) will be given explicitly.

3. Applications

3.1. Bessel functions of first kind

Let \( J_p(z) \) be the Bessel function of first kind of order \( p \)

\[
J_p(z) = \left( \frac{z}{2} \right)^p \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(p + k + 1)} \left( \frac{z}{2} \right)^{2k} .
\]

Then we have the following result

**Theorem 3.1.** The functions \( z^p J_p(z) \) and \( z^{-p} J_p(z) \) satisfy the following differential equations

\[
\left[ z \frac{d^2}{dz^2} + (1 - 2p) \frac{d}{dz} + z \right] (z^p J_p(z)) = 0 , \tag{3.1}
\]
\[
\left[ z \frac{d^2}{dz^2} + (1 + 2p) \frac{d}{dz} - z \right] (z^{-p} J_p(z)) = 0 \quad (3.2)
\]

respectively.

**Proof.** First we consider the function \( z^p J_p(z) \). Since it is easy to verify the relation
\[
\frac{1}{z} \frac{d}{dz} [z^p J_p(z)] = z^{p-1} J_{p-1}(z),
\]
we can define the operator \( E_p^- \) in the following way
\[
E_p^- = \frac{1}{z} \frac{d}{dz}.
\]
Furthermore they satisfy the recurrence relation
\[
2p(z^p J_p(z)) = z^{2} (z^{p-1} J_{p-1}(z)) + z^{p+1} J_{p+1}(z).
\]
By using the Theorem 2.1 we can write
\[
E_p^+ = 2p - z \frac{d}{dz}.
\]
Then the eq. (3.1) immediately follows from the factorization (2.3).

In an analogous way, for \( z^{-p} J_p(z) \), we find the operator
\[
E_p^+ = \frac{1}{z} \frac{d}{dz}.
\]
Furthermore they satisfy the recurrence relation
\[
2p(z^{-p} J_p(z)) = z^{-(p-1)} J_{p-1}(z) + z^{2} (z^{-(p+1)} J_{p+1}(z)).
\]
By using the Theorem 2.1 we can write
\[
E_p^- = 2p + z \frac{d}{dz}.
\]
Then the eq. (3.2) immediately follows from the factorization (2.3).

**Remark 3.1.** Since we will use always the same method, which is based on Theorem 2.1, in the following we will limit ourselves to derive explicitly the operators \( E_p^- \) and \( E_p^+ \) by means of which the searched differential operator can be factorized.
3.2. Bessel functions of second kind

Let \( Y_p(z) \) be the Bessel function of second kind of order \( p \)
\[
Y_p(z) = J_p(z) \cos(p\pi) - (-1)^p J_p(z) \sin(p\pi).
\]

Then we have the following result

**Theorem 3.2.** The function \( z^p Y_p(z) \) satisfies the following differential equation
\[
\left[ z \frac{d^2}{dz^2} + (1 - 2p) \frac{d}{dz} + z \right] (z^p Y_p(z)) = 0.
\] (3.3)

**Proof.** By a similar procedure, we can derive Eq. (3.3) from the factorization (2.3) where
\[
E_p^- = \frac{1}{z} \frac{d}{dz},
\]
and
\[
E_p^+ = 2p - z \frac{d}{dz}.
\]

3.3. Bessel functions of third kind

Let \( H^{(1)}_p(z) \) be the Bessel function of third kind (or Henkel function) of order \( p \)
\[
H^{(1)}_p(z) = J_p(z) + i Y_p(z).
\]

Then we have the following result.

**Theorem 3.3.** The function \( z^p H^{(1)}_p(z) \) satisfies the following differential equation
\[
\left[ z \frac{d^2}{dz^2} + (1 + 2p) \frac{d}{dz} + z \right] (z^p H^{(1)}_p(z)) = 0.
\] (3.4)

**Proof.** We find, in this case:
\[
E_p^- = \frac{1}{z} \frac{d}{dz},
\]
and
\[
E_p^+ = 2p - z \frac{d}{dz}.
\]
Then the eq. (3.4) immediately follows from the factorization (2.3).

3.4. Modified Laguerre polynomials

These polynomials are defined by the generating function (see [13], Eq. 8.4 (46), p.425)

\[ (1 - t)^{-\alpha} e^{xt} = \sum_{n=0}^{\infty} f_n^{\alpha}(x) t^n = G_\alpha(x, t), \]

and are connected with the classical Laguerre polynomials \( L_n^{(\alpha)}(x) \) or the Poisson-Charlier polynomials \( c_n(x; \alpha) \) by means of the formula (see [13], Eq. 8.4 (45), p.425)

\[ f_n^{-\alpha}(x) = (-1)^n L_n^{(\alpha-n)}(x) = \frac{x^n}{n!} c_n(\alpha; x). \] (3.5)

The explicit expression for these functions is given by

\[ f_n^{\alpha}(x) = \sum_{k=0}^{n} \frac{x^k \Gamma(n - k + \alpha)}{k! (n - k)! \Gamma(\alpha)}, \]

while their integral representation is given by

\[ f_n^{\alpha}(x) = \frac{1}{n! \Gamma(\alpha)} \int_{0}^{\infty} e^{-\xi} \xi^{\alpha-1}(x + \xi)^n d\xi. \]

We can prove the following result.

**Theorem 3.4.** The modified Laguerre polynomials satisfy the following differential equation

\[ \left[ x \frac{d^2}{dx^2} - (x + \alpha) \frac{d}{dx} + 1 \right] f_n^{\alpha}(x) = 0. \] (3.6)

**Proof.** Since these polynomials belong to the Appell class, we have

\[ E_n^{-} = \frac{d}{dx}. \]

Furthermore they satisfy the recurrence relation

\[
\begin{align*}
  f_1^{\alpha}(x) &= (x + \alpha) f_0^{\alpha}(x), & (n = 0), \\
  f_{n+1}^{\alpha}(x) &= \frac{1}{n+1} [n + (x + \alpha)] f_n^{\alpha}(x) - x f_{n-1}^{\alpha}(x), & (n = 1, 2, \ldots).
\end{align*}
\]
By using the Theorem 2.1 we can write
\[ E_n^+ = \frac{1}{n+1} \left[ n + (x + \alpha) - x \frac{d}{dx} \right]. \]

**Remark 3.2.** Note that using formula (3.5) we can rewrite Eq. (3.6) in terms of the more familiar \( L \)-notation of the classical Laguerre polynomials, that is,
\[ \left[ x \frac{d^2}{dx^2} - (x - \alpha) \frac{d}{dx} + 1 \right] L_n^{(\alpha-n)}(x) = 0. \]

### 3.5. Generalized Hermite-type truncated exponential polynomials

Starting by the (Gould-Hopper) generalized Hermite polynomials (see [13], Eq. 1.9 (6), p.76)
\[ g_m^n(x, h) = \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{n!}{k! \left( n - mk \right)!} h^k x^{n-mk}, \]
where \( m \) is a positive integer, we can define the following polynomials
\[ [m] h_n(x) = \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{x^{n-mk}}{(n-mk)!}, \]
called generalized Hermite-type truncated exponential polynomials.

For these polynomials we have the integral representation
\[ [m] h_n(x) = \frac{1}{n!} \int_0^\infty e^{-\xi} g_m^n(x, \xi) d\xi, \]
and, by means of the formula
\[ \sum_{n=0}^\infty \frac{t^n}{n!} g_m^n(x, h) = e^{xt} h^m, \]
we obtain the generating function of \([m] h_n(x)\)
\[ \sum_{n=0}^\infty t^n ([m] h_n(x)) = \frac{e^{xt}}{1 - t^m} = G_m(x, t). \]

We can prove the following result.
Theorem 3.5. The Generalized Hermite-type truncated exponential polynomials satisfy the following differential equation

\[ \left( x \frac{d^{m+1}}{dx^{m+1}} - n \frac{d^{m}}{dx^{m}} + x \frac{d}{dx} + n \right) [m] h_n(x) = 0. \]  

(3.7)

Proof. In this case the operator \( E_n^- \) is given by

\[ E_n^- = \frac{d}{dx}, \]

while the recurrence relation is given by

\[ [m] h_{n+1}(x) = \frac{1}{n+1} \left( (n+1)[m] h_{n+1-m}(x) + x[m] h_{n}(x) - x[m] h_{n-m}(x) \right). \]

By using the Theorem 2.1 we can write

\[ E_n^+ = \frac{d^{m-1}}{dx^{m-1}} + \frac{x}{n+1} \left( 1 - \frac{d^{m}}{dx^{m}} \right). \]

3.6. An extension of the Laguerre-type polynomials

Let \( l_n(x) \) be the so called Laguerre-type truncated exponential polynomials, defined by

\[ l_n(x) = \sum_{k=0}^{n} \frac{(-1)^k x^k}{(k!)^2}. \]

A generalization of these polynomials is given by the following new class of polynomials

\[ l_{n,\alpha}(x) = \sum_{k=0}^{n} \frac{(-1)^k x^k}{k! \Gamma(k + \alpha + 1)}, \]

where \( \alpha \) is a real number.

By means of the associate Laguerre polynomials

\[ L_n^{(\alpha)}(x, y) = \sum_{k=0}^{n} \frac{(-x)^k \Gamma(n + \alpha + 1) y^{n-k}}{(n-k)! k! \Gamma(\alpha + k + 1)}, \]

and their generating function

\[ \sum_{n=0}^{\infty} \frac{t^n}{n! (n + \alpha + 1)} L_n^{(\alpha)}(x, y) = e^{yt} C_{\alpha}(x), \]
where \( C_\alpha(xt) \) is the Tricomi-Bessel function
\[
C_\alpha(xt) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k t^k}{k! \ (\alpha + k)!},
\]
we can write the integral representation
\[
l_{n,\alpha}(x) = \frac{1}{\Gamma(n + \alpha + 1)} \int_0^\infty e^{-\xi} L_n^{(\alpha)}(x, \xi)d\xi,
\]
and the generating function
\[
\sum_{n=0}^{\infty} t^n l_{n,\alpha}(x) = \frac{1}{1 - t C_\alpha(xt)}.
\]

**Theorem 3.6.** The polynomials \( l_{n,\alpha}(x) \) satisfy the following differential equation
\[
\begin{align*}
&\left\{ x^3 \frac{d^4}{dx^4} + x^2 \frac{d^3}{dx^3} (\alpha + 6) + x \frac{d^2}{dx^2} [x + \alpha + 1 + (\alpha + 1)(\alpha + 2) + 2(\alpha + 2) \\
&- (n + 1)(n + \alpha + 1)] + \frac{d}{dx} [(\alpha + 1)^2 + x(\alpha + 1) - (n + 1)(\alpha + 1)(n + \alpha + 1) + x] \\
&+ (\alpha + 1) + (n + 1)(n + \alpha + 1) \right\} l_{n,\alpha}(x) = 0. \\
&\text{ (3.8)}
\end{align*}
\]

**Proof.** For these polynomials we have
\[
E_n^- = -\frac{1}{x^\alpha} \frac{d}{dx} \frac{d}{dx} x^{\alpha + 1} \frac{d}{dx}.
\]
Furthermore we have the following recurrence relation
\[
l_{n+1,\alpha}(x) = \frac{1}{(n+1)(n + \alpha + 1)} \left( [(n+1)(n + \alpha + 1) - x]l_{n,\alpha}(x) + xl_{n-1,\alpha}(x) \right).
\]
By using the Theorem 2.1 we can write
\[
E_n^+ = 1 - \frac{x}{(n+1)(n + \alpha + 1)} \left( 1 + \frac{1}{x^\alpha} \frac{d}{dx} x^{\alpha + 1} \frac{d}{dx} \right).
\]

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References


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