EXISTENCE AND CONTROLLABILITY RESULTS FOR MULTIVALUED SEMILINEAR DIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS

BY

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Abstract. In this paper, we shall establish sufficient conditions for the existence of solutions for semilinear functional differential inclusions in Banach spaces with nonlocal conditions. We shall rely of a fixed point theorem for contraction multivalued maps due to Covitz and Nadler.

1. Introduction and Preliminaries

In this paper, we shall establish sufficient conditions for the existence of solutions of semilinear functional differential inclusions in Banach spaces with nonlocal initial conditions. More precisely in Section 2 we consider the following delay semilinear differential inclusion of the form

\[ y' - Ay \in F(t, y(\sigma(t))), \quad t \in J := [0, b], \]

\[ y(0) + f(y) = y_0, \]

where \( F : J \times X \to P(X) \) is a multivalued map, \( A \) is the infinitesimal generator of a strongly continuous semigroup \( T(t), t \geq 0 \), \( f : C(J, X) \to X \), \( P(X) \) is the family of all subsets of \( X \), \( X \) is a real separable Banach space with norm \( |\cdot| \), and \( \sigma : J \to J \) a continuous function satisfying \( \sigma(t) \leq t \) for \( t \in J \).
In Section 3 we study semilinear functional differential inclusions of the form

\[ y' - Ay \in F(t, y_t), \quad \text{a.e.} \quad t \in J = [0, b], \]  

\[ y(0) + (\xi(y_{t_1}, \ldots, y_{t_p}))(t) = \phi(t), \quad t \in [-r, 0], \]  

where \( F \) as in problem (1)-(2), \( \phi \in C(J_0, X) \) (here \( J_0 = [-r, 0] \)), \( 0 < t_1 < \ldots < t_p \leq b, p \in \mathbb{N} \) and \( \xi: [C(J_0, X)]^p \to C(J_0, X) \).

For any continuous function \( y \) defined on the interval \( J_1 = [-r, b] \) and any \( t \in J \), we denote by \( y_t \) the element of \( C(J_0, X) \) defined by

\[ y_t(\theta) = y(t + \theta), \quad \theta \in J_0. \]

Here \( y_t(\cdot) \) represents the history of the state from time \( t - r \), up to the present time \( t \).

Section 4 is devoted to the existence of solutions of the following semilinear neutral functional differential inclusions

\[ \frac{d}{dt} [y(t) - h(t, y_t)] \in Ay(t) + F(t, y(t)), \quad t \in J := [0, b], \]  

\[ y(t) + (\xi(y_{t_1}, \ldots, y_{t_p}))(t) = \phi(t), \quad t \in [-r, 0], \]

where \( F, \xi \) are as in problems (1)-(2) and (3)-(4) and \( h: J \times C(J_0, X) \to X \).

In Section 5 we give some applications on controllability results.

The pioneering work on evolution initial value problems, (IVP for short), with nonlocal conditions is due to Byszewski. As pointed out by Byszewski [16], [15] the study of IVP with nonlocal conditions is of significance since they have applications in problems in physics and other areas of applied mathematics. In fact, more authors have paid attention to the research of IVP with nonlocal conditions, in the few past years. We refer to Balachandran and Chandrasekaran [5], Byszewski [15], [16], Ntouyas [27] and Ntouyas and Tsamatos [25], [26]. For problems with multivalued right-hand side see Benchohra and Ntouyas [9]-[13].

However, in all the above works of the authors, the right-hand side was assumed to be convex valued. Here, we drop this restriction, and consider the problems (1)-(2), (3)-(4) and (5)-(6) with a nonconvex valued right hand side. Our analysis relies on a fixed point theorem for contraction multivalued maps due to Covitz and Nadler [19] (see also Deimling, [20] Theorem 11.1).
In the following by $C(J_0, X)$ we denote the Banach space of all continuous functions from $J_0$ into $X$ with the norm
\[ \| \phi \| = \sup \{|\phi(\theta)| : -r \leq \theta \leq 0\} \]
and by $C(J_1, X)$ the Banach space of all continuous functions from $J_1$ into $X$ with the norm
\[ \| y \|_\infty := \sup \{|y(t)| : t \in J_1\}. \]
$B(X)$ denotes the Banach space of bounded linear operators from $X$ into $X$ with norm
\[ \| N \|_{B(X)} := \sup \{|N(y)| : |y| = 1\}. \]
Let $(X, d)$ be a metric space. We use the notations:
\[ P(X) = \{Y \in P(X) : Y \neq \emptyset\}, \quad P_{cl}(X) = \{Y \in P(X) : Y \text{ closed}\}, \quad P_b(X) = \{Y \in P(X) : Y \text{ bounded}\}, \quad P_{op}(X) = \{Y \in P(X) : Y \text{ compact}\} \text{ and } P_c(X) = \{Y \in P(X) : Y \text{ convex}\}. \]
Consider $H_d : P(X) \times P(X) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$, given by
\[ H_d(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b) \right\}, \]
where $d(A, b) = \inf_{a \in A} d(a, b)$, $d(a, B) = \inf_{b \in B} d(a, b)$.

Then $(P_{b,d}(X), H_d)$ is a metric space and $(P_{d}(X), H_d)$ is a generalized metric space.

For details on multivalued maps we refer to the books of Deimling [20], Gorniewicz [22] and Hu and Papageorgiou [24].

2. Delay Semilinear Differential Inclusions

**Definition 2.1.** A function $y \in C(J, X)$ is called a mild solution of (1)–(2) if there exists a function $v \in L^1(J, X)$ such that $v(t) \in F(t, y(\sigma(t)))$ a.e. on $J$, and
\[ y(t) = T(t)y_0 - T(t)f(y) + \int_0^t T(t-s)v(s)ds. \]

We will need the following assumptions:
Now we are able to state and prove our main result for this section.

**Theorem 2.1.** Assume that hypotheses (H1)–(H6) are satisfied. Then the problem (1)–(2) has at least one mild solution on \( J \).

**Proof.** Transform the problem (1)–(2) into a fixed point problem. Consider the multivalued operator \( N : C(J, X) \to P(C(J, X)) \) defined by:

\[
N(y) := \left\{ h \in C(J, X) : h(t) = T(t)(y_0 - f(y)) + \int_0^t T(t - s)g(s)ds : g \in S_{F,y} \right\}
\]

where

\[
g \in S_{F,y} = \{ g \in L^1(J, X) : g(t) \in F(t, y(\sigma(t))) \text{ for a.e. } t \in J \}.
\]

We shall show that \( N \) is a contraction, by applying Covitz and Nadler’s fixed point theorem.

Using (H3) we can easily show that \( N(y) \in P_d(C(J, X)) \) for each \( y \in C(J, X) \). We shall prove that

\[
H_d(N(y_1), N(y_2)) \leq \gamma \|y_1 - y_2\|_{\infty} \text{ for each } y_1, y_2 \in C(J, X) \quad \text{(where } \gamma < 1).\]
Let $y_1, y_2 \in C(J, X)$ and $h_1 \in N(y_1)$. Then there exists $g_1(t) \in F(t, y_1(\sigma(t)))$ such that

$$h_1(t) = T(t)(y_0 - f(y_1)) + \int_0^t T(t - s)g_1(s)ds, \quad t \in J.$$  

From (H3) it follows that

$$H_d(F(t, y_1(\sigma(t))), F(t, y_2(\sigma(t)))) \leq l(t)|y_1(\sigma(t)) - y_2(\sigma(t))| \leq l(t)|y_1(t) - y_2(t)|.$$  

Hence there is $w \in F(t, y_2(\sigma(t)))$ such that

$$|g_1(t) - w| \leq l(t)|y_1(t) - y_2(t)|, \quad t \in J.$$  

Consider $U : J \to P(X)$, given by

$$U(t) = \{ w \in X : |g_1(t) - w| \leq l(t)|y_1(t) - y_2(t)| \}.$$  

Since the multivalued operator $V(t) = U(t) \cap F(t, y_2(\sigma(t)))$ is measurable (see Proposition III.4 in [18]) there exists $g_2(t)$ a measurable selection for $V$. So, $g_2(t) \in F(t, y_2(\sigma(t)))$ and

$$|g_1(t) - g_2(t)| \leq l(t)|y_1(t) - y_2(t)|, \quad \text{for each} \quad t \in J.$$  

Let us define for each $t \in J$

$$h_2(t) = T(t)(y_0 - f(y_2)) + \int_0^t T(t - s)g_2(s)ds.$$  

Then we have

$$|h_1(t) - h_2(t)| \leq M|f(y_1) - f(y_2)| + M\int_0^t |g_1(s) - g_2(s)|ds$$

$$\leq Mc\|y_1 - y_2\|_{\infty} + M\int_0^t l(s)|y_1(s) - y_2(s)|ds$$

$$\leq (Mc + Ml^*)\|y_1 - y_2\|_{\infty}.$$  

Then

$$\|h_1 - h_2\|_{\infty} \leq M(c + l^*)\|y_1 - y_2\|_{\infty}.$$  

By the analogous relation, obtained by interchanging the roles of $y_1$ and $y_2$, it follows that

$$H_d(N(y_1), N(y_2)) \leq M(c + l^*)\|y_1 - y_2\|_{\infty}.$$
Then $N$ is a contraction and thus, by Covitz and Nadler’s fixed point theorem, it has a fixed point $y$, which is a mild solution to (1)–(2).

**Remark 2.1.** Consider the following Bielecki-type norm (see [14]) on $C(J, X)$ defined by

$$
\|y\|_B = \max_{t \in J} \{|y(t)|e^{-\tau L(t)}\},
$$

where $L(t) = \int_0^t l(s)ds$ and $\tau \in \mathbb{R}$. Since

$$
e^{-\tau L(b)} \|y\|_\infty \leq \|y\|_B \leq \|y\|_\infty,
$$

the norms $\|y\|_B$ and $\|y\|_\infty$ are equivalent.

Then we can prove step 2 of Theorem 2.1, i.e. $H_d(N(y_1), N(y_2)) \leq \gamma \|y_1 - y_2\|_B$ for each $y_1, y_2 \in C(J, X)$, where

$$
\gamma = M \left( ce^{\tau L(b)} + \frac{1}{\tau} \right).
$$

Indeed, we have:

$$
\|h_1 - h_2\|_B \leq \max_{t \in J} e^{-\tau L(t)} M|f(y_1) - f(y_2)| + \max_{t \in J} e^{-\tau L(t)} M \int_0^t |g_1(s) - g_2(s)|ds \\
\leq Mce^{\tau L(b)} \|y_1 - y_2\|_B + M \|y_1 - y_2\|_B \max_{t \in J} e^{-\tau L(t)} \int_0^t l(s)e^{\tau L(s)}ds \\
\leq Mce^{\tau L(b)} \|y_1 - y_2\|_B + M \|y_1 - y_2\|_B \frac{1}{\tau}(1 - e^{-\tau L(b)}) \\
\leq M \left( ce^{\tau L(b)} + \frac{1}{\tau} \right) \|y_1 - y_2\|_B.
$$

We can choose $c$ and $\tau$ such that $\gamma < 1$. In this case assumption (H6) must be deleted.

**Example.** Consider the following partial differential equation of the form

$$
z_t(y, t) - z_{yy}(y, t) = q(t, z(y, t)), \quad 0 \leq y \leq \pi, \quad t \in J \tag{7}
$$

$$
z(0, t) = z(\pi, t) = 0, \quad z(y, 0) + z(y, 1) = z_0(y), \tag{8}
$$

where $q : J \times X \to X$, is a given function.

Take $X = L^2[0, \pi]$ and define $A : X \to X$ by $Aw = w''$ with domain

$$
D(A) = \{w \in X, w, w' \text{ are absolutely continuous, } w'' \in X, w(0) = w(\pi) = 0\}.
$$
Then
\[ Aw = \sum_{n=1}^{\infty} n^2(w, w_n)w_n, \quad w \in D(A) \]
where \( w_n(s) = \sqrt{2/\pi} \sin ns, \ n = 1, 2, \ldots \) is the orthogonal set of eigenvectors in \( A \). It is easily shown ([28]) that \( A \) is the infinitesimal generator of a strongly continuous analytic semigroup \( T(t), \ t \geq 0 \) and is given by the formula
\[ T(t)w = \sum_{n=1}^{\infty} e^{-n^2t}(w, w_n)w_n, \quad w \in X. \]

Assume that there exists an integrable function \( p : J \to [0, \infty) \) such that
\[ |q(t, w_1) - q(t, w_2)| \leq p(t)\|w_1 - w_2\| \quad \text{for a.e.} \quad t \in J \quad \text{and for each} \quad w_1, w_2 \in X. \]
Then the problem (1)–(2) is an abstract formulation of the problem (7)–(8). Since all the conditions of Theorem 2.1 are satisfied, the problem (7)–(8) has a solution on \( J \).

3. Semilinear Functional Differential Inclusions

**Theorem 3.1.** Assume that hypotheses (H1) and (H2)–(H3) (with \( y \in C(J_1, X) \)) are satisfied. Assume also that:

(H7) There exists a constant \( K > 0 \) such that
\[ \|(\xi(y_{11}, \ldots, y_{1p}))(t) - (\xi(\tilde{y}_{11}, \ldots, \tilde{y}_{1p}))(t)\| \leq K\|y - \tilde{y}\|_{\infty} \]
for \( y, \tilde{y} \in C(J_1, X), \ t \in J_0. \)

(H8) \( M(K + l^t) < 1 \).

Then the problem (3)–(4) has at least one mild solution on \( J_1 \).

**Proof.** Transform the problem (3)–(4) into a fixed point problem. Consider the multivalued operator \( N \) as defined in Theorem 2.1, with \( \xi \) in place of \( f \).

We shall prove that
\[ H_d(N(y_1), N(y_2)) \leq \gamma\|y_1 - y_2\|_{\infty} \quad \text{for each} \quad y_1, y_2 \in C(J_1, X) \quad \text{(where} \quad \gamma < 1). \]
Let $y_1, y_2 \in C(J_1, X)$ and $h_1 \in N(y_1)$. Then there exists $g_1(t) \in F(t, y_{1t})$ such that

$$h_1(t) = T(t) \left[ \phi(0) - (\xi((y_1)_{t_1}, \ldots, (y_1)_{t_p}))(0) \right] + \int_0^t T(t-s)g_1(s)ds.$$ 

From (H3) it follows that

$$H_d(F(t, y_{1t}), F(t, y_{2t})) \leq l(t)\|y_{1t} - y_{2t}\|.$$

Hence there is $w \in F(t, y_{2t})$ such that

$$|g_1(t) - w| \leq l(t)\|y_{1t} - y_{2t}\|, \quad t \in J.$$

Consider $U : J \to \mathcal{P}(X)$, given by

$$U(t) = \{w \in X : |g_1(t) - w| \leq l(t)\|y_{1t} - y_{2t}\|\}.$$ 

Since the multivalued operator $V(t) = U(t) \cap F(t, y_{2t})$ is measurable (see Proposition III.4 in [18]) there exists $g_2(t)$ a measurable selection for $V$. So, $g_2(t) \in F(t, y_{2t})$ and

$$|g_1(t) - g_2(t)| \leq l(t)\|y_{1t} - y_{2t}\|, \quad \text{for each } t \in J.$$

Let us define for each $t \in J$

$$h_2(t) = T(t) \left[ \phi(0) - (\xi((y_2)_{t_1}, \ldots, (y_2)_{t_p}))(0) \right] + \int_0^t T(t-s)g_2(s)ds.$$ 

Then we have

$$|h_1(t) - h_2(t)| \leq MK\|y_1 - y_2\|_\infty + M\int_0^t l(s)\|y_{1s} - y_{2s}\|ds \leq (MK + ML^*)\|y_1 - y_2\|_\infty.$$

Then

$$\|h_1 - h_2\|_\infty \leq M(K + l^*)\|y_1 - y_2\|_\infty.$$

By the analogous relation, obtained by interchanging the roles of $y_1$ and $y_2$, it follows that

$$H_d(N(y_1), N(y_2)) \leq M(K + l^*)\|y_1 - y_2\|_\infty.$$

Then $N$ is a contraction and thus, it has a fixed point $y$, which is mild solution to (3)–(4).
4. Neutral Semilinear Functional Differential Inclusions

**Definition 4.1.** A function \( y \in C(J_1, X) \) is called a mild solution of (5)--(6) if there exists a function \( v \in L^1(J, X) \) such that \( v(t) \in F(t, y_t) \) a.e. on \( J \), \( y_0 = \phi \), and

\[
y(t) = T(t)[\phi(0)(\xi(y_{t_1}, \ldots, y_{t_p}))(0) - h(0, \phi)] + h(t, y_t) + \int_0^t AT(t-s)h(s, y_s)ds + \int_0^t T(t-s)v(s)ds, \quad t \in J.
\]

**Theorem 4.1.** Assume that (H2)–(H3) (with \( y \in C(J_1, X) \)) and (H7) hold. Further we assume that:

(H9) \( A : D(A) \subset X \to X \) is the infinitesimal generator of a semigroup of bounded linear operators \( T(t) \) in \( E \) such that \( \|T(t)\|_{B(X)} \leq M \), for some \( M > 0 \) and \( \|AT(t)\|_{B(X)} \leq M_0 \), for each \( t > 0 \), for some \( M_0 > 0 \).

(H10) \( |h(t, u) - h(t, \overline{u})| \leq d\|u - \overline{u}\| \), for each \( t \in J \) and \( u, \overline{u} \in C(J_0, X) \), where \( d \) is a nonnegative constant.

Then the problem (5)-6 has at least one mild solution on \( J_1 \), provided \( MK + d + M_0bd + Ml^* < 1 \).

**Proof.** Transform the problem (5)–(6) into a fixed point problem. Consider the multivalued operator \( N_1 : C(J_1, X) \to \mathcal{P}(C(J_1, X)) \) defined by:

\[
N_1(y) := \begin{cases}
\phi(t) - (\xi(y_{t_1}, \ldots, y_{t_p}))(0), & \text{if } t \in J_0 \\
T(t)[\phi(0)(\xi(y_{t_1}, \ldots, y_{t_p}))(0) - h(0, \phi)] + h(t, y_t) + \int_0^t AT(t-s)h(s, y_s)ds + \int_0^t T(t-s)v(s)ds, & \text{if } t \in J.
\end{cases}
\]

Clearly the operator \( N_1 \) is closed valued. We will prove that \( H_d(N_1(y_1), N_1(y_2)) \leq \tilde{\gamma}\|y_1 - y_2\|_\infty \) for each \( y_1, y_2 \in C(J_1, X) \) (where \( \tilde{\gamma} < 1 \)).
Let $y_1, y_2 \in C(J_1, X)$ and $h_1 \in N(y_1)$. Then there exists $g_1(t) \in F(t, y_{1t})$ such that
\[
h_1(t) = T(t)[\phi(0) - (\xi((y_1)_{t_1}, \ldots, (y_1)_{t_p}))(0) - h(0, \phi)] + h(t, y_{1t})
+ \int_0^t A(T(t - s)h(s, y_{1s}) + \int_0^t T(t - s)g_1(s)ds, \quad t \in J.
\]
From (H3) it follows that
\[
H_d(F(t, y_{1t}), F(t, y_{2t})) \leq l(t)\|y_{1t} - y_{2t}\|.
\]
Hence there is $w \in F(t, y_{2t})$ such that
\[
|g_1(t) - w| \leq l(t)\|y_{1t} - y_{2t}\|, \quad t \in J.
\]
Consider $U : J \to P(X)$, given by
\[
U(t) = \{w \in X : |g_1(t) - w| \leq l(t)\|y_{1t} - y_{2t}\|\}.
\]
Since the multivalued operator $V(t) = U(t) \cap F(t, y_{2t})$ is measurable (see Proposition III.4 in [18]) there exists $g_2(t)$ a measurable selection for $V$. So, $g_2(t) \in F(t, y_{2t})$ and
\[
|g_1(t) - g_2(t)| \leq l(t)\|y_{1t} - y_{2t}\|, \quad \text{for each} \quad t \in J.
\]
Let us define for each $t \in J$
\[
h_2(t) = T(t)[\phi(0) - (\xi((y_2)_{t_1}, \ldots, (y_2)_{t_p}))(0) - h(0, \phi)] + h(t, y_{2t})
+ \int_0^t A(T(t - s)h(s, y_{2s}) + \int_0^t T(t - s)g_2(s)ds, \quad t \in J.
\]
Then we have
\[
|h_1(t) - h_2(t)| \leq MK\|y_1 - y_2\|_\infty + d\|y_{1t} - y_{2t}\| + M_0d \int_0^t \|y_{1s} - y_{2s}\|ds
+ M \int_0^t l(s)\|y_{1s} - y_{2s}\|ds
\leq [MK + d + M_0bd + ML^*]\|y_1 - y_2\|_\infty.
\]
Then
\[
\|h_1 - h_2\|_\infty \leq [MK + d + M_0bc + ML^*]\|y_1 - y_2\|_\infty.
\]
By the analogous relation, obtained by interchanging the roles of \( y_1 \) and \( y_2 \), it follows that
\[
H_d(N_1(y_1), N_1(y_2)) \leq [MK + d + M_0bd + Ml^*] \| y_1 - y_2 \|_\infty.
\]
Then \( N_1 \) is a contraction and thus, it has a fixed point \( y \), which is a mild solution to (5)–(6).

5. Applications

As applications of the results of the previous sections we give in this one some controllability results for functional differential inclusions. For recent controllability results of nonlinear ordinary, functional and neutral functional differential and integrodifferential systems in Banach spaces, by using different tools of fixed point arguments we refer to the papers by Benchohra and Ntouyas [7] and Balachandran et al. [1], [2] and [3], respectively. Very Recently, by means of a fixed point theorem for condensing multivalued maps due to Martelli, we have studied in [6] and [8] the controllability of second order multivalued semilinear differential and integrodifferential equations with a convex valued right-hand side.

In this section we consider first the system (3)–(4) with a control parameter such as
\[
y' - Ay \in F(t, y_t) + (Bu)(t), \quad t \in J, \tag{9}
y(t) + (\xi(y_{t_1}, \ldots, y_{t_p}))(t) = \phi(t), \quad t \in J_0 \tag{10}
\]
where \( B \) is a bounded linear operator from \( U \), a Banach space, to \( X \) and \( u \in L^2(J, U) \).

**Definition 5.1.** The system (9)–(10) is said to be nonlocally controllable on the interval \( J_1 \), if for every continuous initial function \( \phi \in C \) and every \( y_1 \in X \) there exists a control \( u \in L^2(J, U) \), such that the mild solution \( y(t) \) of (9)–(10) satisfies \( y(b) + (\xi(y_{t_1}, \ldots, y_{t_p}))(b) = y_1 \).

To establish the controllability result we need the following additional hypotheses:
(H11) the linear operator $W : L^2(J, U) \rightarrow X$, defined by

$$Ww = \int_0^b T(b - s)Bw(s)\,ds,$$

has an invertible operator $W^{-1}$ which takes values in $L^2(J, U)/\ker W$ and there exist positive constants $M_1$ and $M_2$ such that $|B| \leq M_1$ and $|W^{-1}| \leq M_2$.

**Remark 5.1.** Examples with $W : L^2(J, U) \rightarrow X$ such that $W^{-1}$ exists and is bounded are discussed in [17].

**Theorem 5.1.** Assume that hypotheses (H1) and (H2)–(H3) (with $y \in C(J_1, X)$) and (H7), (H11) are satisfied. Then the system (9)–(10) is nonlocally controllable on the interval $J_1$.

**Proof.** Using hypothesis (H11) for an arbitrary function $y(\cdot)$ define the control

$$u_y(t) = W^{-1}\left[y_1 - (\xi(y_{t_1}, \ldots, y_{t_p}))(b) - T(b)\phi(0) + T(b)(\xi(y_{t_1}, \ldots, y_{t_p}))(b) - \int_0^b T(b - s)g(s)\,ds\right](t), \quad g \in S_{F,y}.$$

We shall now show that when using this control, the multivalued map, $N_2 : C(J_1, X) \rightarrow \mathcal{P}(C(J_1, X))$ defined by:

$$N_2(y) := \begin{cases} h \in C(J_1, X) : h(t) = \begin{cases} \phi(t) - (\xi(y_{t_1}, \ldots, y_{t_p}))(t), & \text{if } t \in J_0 \\ T(t)[\phi(0) - (\xi(y_{t_1}, \ldots, y_{t_p}))(0)] + \int_0^t T(t - s)(Bu_y)(s)\,ds + \int_0^t T(t - s)g(s)\,ds, & \text{if } t \in J \end{cases} \end{cases}$$

has a fixed point. This fixed point is then a solution of the system (9)–(10). We omit the details.

By the same method we can study controllability of the neutral functional differential inclusions

$$\frac{d}{dt}[y(t) - h(t, y_0)] \in Ay(t) + F(t, y_t) + (Bu)(t), \quad t \in J,$$

(11)
\[ y(t) + (\xi(y_{t_1}, \ldots, y_{t_p}))(t) = \phi(t), \quad t \in J_0. \] (12)

The control \( u(.) \) is given by

\[
u_y(t) = W^{-1} \left[ y_1 - T(b)[\phi(0) - \xi(y_{t_1}, \ldots, y_{t_p})(0) - h(0, \phi)] - h(b, y_b) \right.
\]
\[
- \int_0^b AT(b - s)h(s, y_s)ds - \int_0^b T(b - s)g(s)ds \right](t), \quad g \in SF_y.
\]

**Remark 5.2.** The reasoning used above can be used for the existence and the controllability result of the corresponding semilinear functional and neutral functional integrodifferential inclusions of Volterra type.

**References**


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