ON SOME SPECIAL CONDITIONS OF OVER-DAMPED NONLINEAR SYSTEMS

BY

M. SHAMSUL ALAM

Abstract. The method of Krylov-Bogoliubov-Mitropolskii is extended to certain over-damped nonlinear systems. The damping force is considered large or very large. The method is illustrated by an example.

1. Introduction

Krylov-Bogoliubov-Mitropolskii (KBM) method [1, 2] is well known in the theory of nonlinear vibrations. The method was originally developed by Krylov and Bogoliubov [1] for obtaining the periodic solution of nonlinear systems with small nonlinearities. Then the method was amplified and justified by Bogoliubov and Mitropolskii [2, 3, 4, 5]. Murty, Deekshatulu and Krisna [6] extended the method to over-damped nonlinear systems, but one can not use Murty, Deekshatulu and Krisna’s [6] solution for some damping forces, especially when one of the characteristic roots of the corresponding linear systems is double or triple of another. Recently, author [7] has investigated over-damped nonlinear systems and has been able to find two solutions of Duffing’s equation when one root is respectively double or triple of the other. However, the solutions obtained in [6] and [7] are unable to give desired results when one root becomes much smaller than the other root. In this paper, approximate solutions of such over-damped nonlinear systems have been found. The solutions show good coincidence with numerical solutions.
2. Method

Consider a second order nonlinear system governed by the ordinary differential equation

\[ \ddot{x} + c_1 \dot{x} + c_2 x = -\varepsilon f(x, \dot{x}), \]  

where the over-dots denote differentiation with respect to \( t \), \( \varepsilon \) is a small parameter, \( c_1 \) and \( c_2 \) are constants, and \( f \) is the given nonlinear function. When \( c_1 > \sqrt{c_2} \), the characteristic roots of the linear equation of (1) are real and unequal say, \(-\lambda - \mu, \lambda > \mu > 0\), so that (1) represents an over-damped system. Therefore, the linear over-damped solution is

\[ x(t, 0) = a_0 e^{-\lambda t} + b_0 e^{-\mu t}, \]

where \( a_0 \) and \( b_0 \) are arbitrary constants. We investigate the above nonlinear system when the linear damping force, \(-2c_1 \dot{x}\), is large or very large, i.e., \( c_1 >> \sqrt{c_2} \) or \( \lambda >> \mu \).

We choose an approximate solution of (1) in the form

\[ x(t, \varepsilon) = a(t)e^{-\lambda t} + b(t)e^{-\mu t} + \varepsilon u_1(a, b, t) + \varepsilon^2 u_2(a, b, t) + \varepsilon^3 \cdots, \]

where \( a \) and \( b \) satisfy the differential equations

\[ \dot{a} = \varepsilon A_1(a, b, t) + \varepsilon^2 A_2(a, b, t) + \varepsilon^3 \cdots, \]

\[ \dot{b} = \varepsilon B_1(a, b, t) + \varepsilon^2 B_2(a, b, t) + \varepsilon^3 \cdots. \]

Confining only to the first few terms, \( 1, 2, \cdots, m \) in the series expansions of (3) and (4), we evaluate the functions \( u_1, u_2, \cdots \) and \( A_1, A_2, \cdots; B_1, B_2, \cdots \) such that \( a(t) \) and \( b(t) \) appearing in (3) and (4) satisfy the given differential equation (1) with an accuracy of \( \varepsilon^{m+1} \). In order to determine these unknown functions it is assumed that the functions \( u_1, u_2, \cdots \) do not contain some secular-type terms ([7]).

Now differentiating (3) twice with respect to \( t \) and utilizing relation (4), we obtain

\[ \dot{x} = -\lambda a e^{-\lambda t} - \mu b e^{-\mu t} + \varepsilon \left( e^{-\lambda t} A_1 + e^{-\mu t} B_1 + \frac{\partial u_1}{\partial t} \right) + \varepsilon^2 \cdots, \]

\[ \ddot{x} = \lambda^2 a e^{-\lambda t} + \mu^2 b e^{-\mu t} + \varepsilon \left( e^{-\lambda t} \left( \frac{\partial}{\partial t} + \lambda \right) A_1 + e^{-\mu t} \left( \frac{\partial}{\partial t} + \mu \right) B_1 + \frac{\partial^2 u_1}{\partial t^2} \right) + \varepsilon^2 \cdots. \]
Substituting the values of $\dot{x}, \ddot{x}$ from (5) and $x$ from (3) into the original equation (1), and comparing the coefficients of $\varepsilon$, we obtain

$$e^{-\lambda t} \left( \frac{\partial}{\partial t} - \lambda + \mu \right) A_1 + e^{-\mu t} \left( \frac{\partial}{\partial t} + \lambda - \mu \right) B_1 + e^{-\lambda t} \left( \frac{\partial}{\partial t} + \lambda \right) e^{-\mu t} u_1 = -f^{(0)}(a, b, t), \quad (6)$$

where $f^{(0)}(a, b, t) = f(x_0, \dot{x}_0)$ and $x_0 = a(t)e^{-\lambda t} + b(t)e^{-\mu t}$.

In general, $f^{(0)}$ be expanded in a Taylor’s series

$$f^{(0)} = \sum_{j,k=0} F_{j,k}(a, b) e^{-(j\lambda + k\mu)t}. \quad (7)$$

It was early imposed by Krylov and Bogoliubov [1] that $u_1$ does not contain secular-terms, $t \cos t$ and $t \sin t$ for obtaining a periodic solution of (1). In order to determine over-damped solutions of (1), author [7] proposed that $u_1$ excludes some of the terms of $f^{(0)}$, namely $e^{-(j\lambda + k\mu)t}$ where $j\lambda + k\mu < c_1(j + k), j = 1, 0; k = 1, 2, \ldots$. In this paper we also follow this assumption. Thus $u_1$ never contain secular-type terms $te^{-k\mu t}$, even $\lambda = 2\mu, 3\mu, \ldots$ or $\mu = 0$. It should be noted that in accordance to Murty, Deekshatulu and Krisna’s [6] assumptions, the perturbation solution sometimes contains secular-type terms $te^{-k\mu t}$ (see [7] for details). Moreover, in order to solve equation (6) for the unknown functions $A_1, B_1$ and $u_1$, we impose a new restriction that $u_1$ excludes terms $e^{-k\mu t}, k = 1, 2, \ldots$ when $\varepsilon = O(1)$. Theoretically, KBM solutions are useful when $\varepsilon << 1$, but Mendelson [8] proved that a damped solution sometimes gives desired results even $\varepsilon = 1$. It is noted that an approximate solution, which can be found under the restriction that $u_1$ excludes terms $e^{-(j\lambda + k\mu)t}, j\lambda + k\mu < c_1(j + k)$, is useless when $\varepsilon = O(1)$, but the solution shows a good agreement with numerical solution when $\varepsilon << 1$. On the contrary, the solution which is found under the restriction that $u_1$ excludes terms $e^{-k\mu t}, k = 1, 2, \ldots$ only, is useful for all values of $\varepsilon \leq 1$. The later solution only agrees with numerical solution while the former shows a good coincidence with the numerical results.

3. Example

As an example of the above procedure, we may consider Duffing’s equation with a large linear damping force, $-2c_1\dot{x}, c_1 >> 1$,

$$\ddot{x} + 2c_1\dot{x} + x = -\varepsilon x^3. \quad (8)$$
Here $\lambda + \mu = 2c_1$, $\lambda\mu = c_2 = 1$. Therefore, we obtain $\lambda > c_1 > 1$ and $\mu < c_1 < 1$, since it has already been considered that $\lambda > \mu$. The function $f^{(0)}$ becomes

$$f^{(0)} = a^3 e^{-3\lambda t} + 3a^2 b e^{-(2\lambda + \mu)t} + 3ab^2 e^{-(\lambda + \mu)t} + b^3 e^{-3\mu t}. \quad (9)$$

It is obvious that $3\lambda > 3c_1$, $2\lambda + \mu > \lambda + 2c_1 > 3c_1$, $\lambda + 2\mu < 2c_1 + \mu < 3c_1$ and $3\mu < 3c_1$, so that $u_1$ excludes terms $e^{-(\lambda + 2\mu)t}$ and $e^{-3\mu t}$ (see [7]). Thus we obtain the following equations

$$e^{-\lambda t} \left( \frac{\partial}{\partial t} - \lambda + \mu \right) A_1 + e^{-\mu t} \left( \frac{\partial}{\partial t} + \lambda - \mu \right) B_1 = -(3ab^2 e^{-(\lambda + 2\mu)t} + b^3 e^{-3\mu t}), \quad (10)$$

and

$$\left( \frac{\partial}{\partial t} + \lambda \right) \left( \frac{\partial}{\partial t} + \mu \right) u_1 = -(a^3 e^{-3\lambda t} + 3a^2 b e^{-(2\lambda + \mu)t}). \quad (11)$$

Solution of (11) is

$$u_1 = - \left( \frac{a^3 e^{-3\lambda t}}{2\lambda(3\lambda - \mu)} + \frac{3a^2 b e^{-(2\lambda + \mu)t}}{2\lambda(\lambda + \mu)} \right). \quad (12)$$

Now we have to determine two functions $A_1$ and $B_1$ from a single equation (10). Author [7] solved (10) when $\lambda \approx 3\mu$ and $\lambda \approx 2\mu$. In this paper, we have considered that the damping force is large or very large, so that $\lambda >> \mu$ or $\lambda + 2\mu \approx \lambda$. Therefore, we can equate the coefficients of $e^{-3\lambda t}$ and $e^{-\mu t}$ from both sides of (10) and obtain the following equations

$$\frac{\partial A_1}{\partial t} - \lambda A_1 + \mu A_1 = -3ab^2 e^{-2\mu t}, \quad (13)$$

and

$$\frac{\partial B_1}{\partial t} + \lambda B_1 - \mu B_1 = -b^3 e^{-2\mu t}. \quad (14)$$

The solution of (13)-(14) is

$$A_1 = \frac{3ab^2 e^{-2\mu t}}{\lambda + \mu}, \quad B_1 = \frac{b^3 e^{-2\mu t}}{3\mu - \lambda}. \quad (15)$$

From the functional relation of $B_1$, it is clear that the solution is useful when $3\mu < \lambda$. It is obvious that as the limit $\lambda \rightarrow 3\mu$, $B_1$ contains a term $te^{-2\mu t}$ (which is not a desired functional relation of $B_1$ in accordance to the theory of perturbation method). Actually, the solution is useful when $3\mu << \lambda$ only (see
[7] for details). Now substituting the values of $A_1$ and $B_1$ from (15) into (4) and then integrating with respect to $t$, we obtain

$$a = a_0 + \frac{3\varepsilon a_0 b_0^2 (1 - e^{-2\mu t})}{2\mu(\lambda + \mu)}, \quad b = \frac{b_0}{\sqrt{1 + \frac{\varepsilon b_0^2 (e^{-2\mu t} - 1)}{\mu(3\mu - \lambda)}}}. \quad (16)$$

Therefore, the first order solution of (8) is

$$x(t, \varepsilon) = a(t)e^{-\lambda t} + b(t)e^{-\mu t} + \varepsilon u_1, \quad (17)$$

where $a$ and $b$ are given by (16) and $u_1$ is given by (12).

It has already been mentioned that when $\varepsilon = O(1)$, $u_1$ excludes terms $e^{-k\mu t}, k = 1, 2, \ldots$. Therefore, $u_1$ excludes only a term $e^{-3\mu t}$ of $f^{(0)}$ and we can rewrite (11), (13) and (14) as :

$$\left(\frac{\partial}{\partial t} + \lambda\right)\left(\frac{\partial}{\partial t} + \mu\right)u_1 = -(a^3 e^{-3\lambda t} + 3a^2 b e^{-(2\lambda+\mu)t} + 3ab^2 e^{-(\lambda+2\mu)t}), \quad (18)$$

and

$$\frac{\partial A_1}{\partial t} - \lambda A_1 + \mu A_1 = 0, \quad (19)$$

and

$$\frac{\partial B_1}{\partial t} + \lambda B_1 - \mu B_1 = -b^3 e^{-2\mu t}. \quad (20)$$

The solution of (18)–(20) is

$$u_1 = -\left(\frac{a^3 e^{-3\lambda t}}{2\lambda(3\lambda - \mu)} + \frac{3a^2 b e^{-(2\lambda+\mu)t}}{2\lambda(\lambda + \mu)} + \frac{3ab^2 e^{-(\lambda+2\mu)t}}{2\mu(\lambda + \mu)}\right), \quad (21)$$

and

$$A_1 = 0, \quad B_1 = \frac{b^3 e^{-2\mu t}}{3\mu - \lambda}. \quad (22)$$

In this case, the solution of (4) is

$$a = a_0, \quad b = \frac{b_0}{\sqrt{1 + \frac{\varepsilon b_0^2 (e^{-2\mu t} - 1)}{\mu(3\mu - \lambda)}}}. \quad (23)$$

4. Discussion of the Author's Previous Solution

In Sec. 3, we have noted that the solution (17) is not useful when $\lambda = 3\mu$. Author [7] previously solved equation (10) for two unknown functions $A_1$ and $B_1$. 
when $\lambda \approx 3\mu$ and $\lambda \approx 2\mu$. When $\lambda \approx 3\mu$, (10) was separated into two following equations

$$e^{-\lambda t} \left( \frac{\partial}{\partial t} - \lambda + \mu \right) A_1 = -b^3 e^{-3\mu t},$$

(24)

and

$$e^{-\mu t} \left( \frac{\partial}{\partial t} + \lambda - \mu \right) B_1 = -3ab^2 e^{-(\lambda+2\mu)t}.$$  

(25)

Therefore, $A_1$ and $B_1$ become

$$A_1 = \frac{b^3 e^{(\lambda-3\mu)t}}{2\mu}, \quad B_1 = \frac{3ab^2 e^{-(\lambda+\mu)t}}{2\mu}.$$  

(26)

Thus $B_1$ does not contain a term $te^{-2\mu t}$, $\mu > 0$. However, the above functional relations of $A_1$ and $B_1$ are useful when $\lambda \approx 3\mu$ only. Author [7] found that the perturbation solution together with $A_1$ and $B_1$ obtained in (26) gives sometime incorrect results when $\lambda \leq 2\mu$. In this case, author [7] further investigated (10) and found that the perturbation solution again shows a good coincidence if (10) is separated into two equations for $A_1$ and $B_1$ as:

$$e^{-\lambda t} \left( \frac{\partial}{\partial t} - \lambda + \mu \right) A_1 = -3ab^2 e^{-(\lambda+2\mu)t} - b^3 e^{-3\mu t},$$

(27)

and

$$e^{-\mu t} \left( \frac{\partial}{\partial t} + \lambda - \mu \right) B_1 = 0.$$  

(28)

The solutions of (27) and (28) are

$$A_1 = \frac{3ab^2 e^{-2\mu t}}{\lambda + \mu} + \frac{b^3 e^{(\lambda-3\mu)t}}{2\mu}, \quad B_1 = 0.$$  

(29)

In this case, the functional relation of $B_1$ is again invalid if $\mu$ is small. Thus author’s [7] previous solutions are useless if $\mu$ is small. However, the perturbation solution together with $A_1$ and $B_1$ obtained in (29) gives desired results until $\lambda - \mu = O(\sqrt{\varepsilon})$.

Thus the solution of (4) becomes

$$a = \begin{cases} 
  a_0 + \frac{\varepsilon b_0^3 (1 - e^{(\lambda-3\mu)t})}{2\mu(-\lambda + 3\mu)}, & \lambda \neq 3\mu, \\
  a_0 + \frac{\varepsilon b_0^3}{2\mu}, & \lambda = 3\mu, 
\end{cases}$$  

(30)

and

$$b = b_0 + \frac{3\varepsilon a_0 b_0^2 (1 - e^{-(\lambda+\mu)t})}{2\mu(\lambda + \mu)},$$
and

\[
a = a_0 + \frac{3\varepsilon a_0 b_0^2 (1 - e^{-2\mu t})}{2\mu(\lambda + \mu)} + \frac{\varepsilon b_0^3 (1 - e^{(\lambda - 3\mu)t})}{2\mu(-\lambda + 3\mu)},
\]

\[
b = b_0,
\]

respectively for \(\lambda \approx 3\mu\) and \(\lambda \approx 2\mu\).

When \(\varepsilon\) is small, (17) represents an over-damped solution of (8), though the solution of (4) is different for various damping forces. However, in all the cases \(u_1\) has remained unchanged. Moreover, (17) represents an over-damped solution of (8) when \(\lambda >> \mu\) and \(\varepsilon = O(1)\).

5. Results and Discussion

On some special conditions, approximate solutions of second-order over-damped nonlinear systems have been found based again on the KBM method. Depending on the linear damping force, \(-2c_1 \dot{x}, c_1 > 1\), the equation (4) has different solutions. In author’s previous paper [7], two different solutions of (4) were found for the same equation (8) (i) when \(\lambda \approx 3\mu\) and (ii) when \(\lambda \approx 2\mu\). In the present paper, a third solution of (4) has been found when \(\lambda >> \mu\). Moreover, a fourth solution of (4) has been found \(\lambda >> \mu\) and \(\varepsilon = O(1)\).

In order to test the accuracy of an approximate solution obtained by a certain perturbation method, we compare the approximate solution to the numerical solution (considered to be exact). With regard to such a comparison concerning the presented KBM method of this paper, we refer author’s [7] previous work. In the present paper, we have compared the perturbation solution (17) to those obtained by Runge-Kutta (fourth-order) method for different values of \(\lambda, \mu\) and \(\varepsilon\).

First of all, \(x(t, \varepsilon)\) has been calculated by (17) in which \(a\) and \(b\) are evaluated by (16) with initial conditions \([x(0) = 1, \dot{x}(0) = 0]\) for \(c_1 = \frac{11}{2\sqrt{10}}\) or \(\lambda = 10\mu = \sqrt{10}\) and \(\varepsilon = 0.1\). Then corresponding numerical solution of (8) has been computed by Runge-Kutta method and percentage errors are calculated. All the results are given in Table 1. From Table 1, it is clear that percentage errors are much smaller than 1%, and thus (17) shows a good coincidence with
the numerical solution. However, increasing with the damping force, the error(s) decreases. To clarify this matter, for $c_1 = 4.25$ or $\lambda = 16\mu = 4$ and $\varepsilon = 0.1$, $x(t, \varepsilon)$ has again been calculated by (17) and correspond numerical solution has been computed. These results are given in Table 2 (with percentage errors). Comparing the results of Table 1 and Table 2, we see that percentage errors of Table 2 are smaller than those errors of Table 1.

Table 1

<table>
<thead>
<tr>
<th>$t$</th>
<th>$x$</th>
<th>$x_{nu}$</th>
<th>$E$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.00000</td>
<td>1.00000</td>
<td>0.0000</td>
</tr>
<tr>
<td>1.0</td>
<td>0.790426</td>
<td>0.789256</td>
<td>0.1482</td>
</tr>
<tr>
<td>2.0</td>
<td>0.568549</td>
<td>0.567283</td>
<td>0.2232</td>
</tr>
<tr>
<td>3.0</td>
<td>0.410185</td>
<td>0.409195</td>
<td>0.2419</td>
</tr>
<tr>
<td>5.0</td>
<td>0.216076</td>
<td>0.215546</td>
<td>0.2459</td>
</tr>
<tr>
<td>7.0</td>
<td>0.114524</td>
<td>0.114244</td>
<td>0.2451</td>
</tr>
<tr>
<td>10.0</td>
<td>0.044314</td>
<td>0.044296</td>
<td>0.2443</td>
</tr>
</tbody>
</table>

Table 2

<table>
<thead>
<tr>
<th>$t$</th>
<th>$x$</th>
<th>$x_{nu}$</th>
<th>$E$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.000000</td>
<td>1.000000</td>
<td>0.0000</td>
</tr>
<tr>
<td>1.0</td>
<td>0.8157880</td>
<td>0.8153834</td>
<td>0.0496</td>
</tr>
<tr>
<td>2.0</td>
<td>0.6264477</td>
<td>0.6259693</td>
<td>0.0764</td>
</tr>
<tr>
<td>3.0</td>
<td>0.4833286</td>
<td>0.4829175</td>
<td>0.0851</td>
</tr>
<tr>
<td>5.0</td>
<td>0.2905257</td>
<td>0.2902690</td>
<td>0.0884</td>
</tr>
<tr>
<td>7.0</td>
<td>0.1756370</td>
<td>0.1754817</td>
<td>0.0885</td>
</tr>
<tr>
<td>10.0</td>
<td>0.0828430</td>
<td>0.0827699</td>
<td>0.0883</td>
</tr>
<tr>
<td>15.0</td>
<td>0.0237257</td>
<td>0.0237048</td>
<td>0.0882</td>
</tr>
</tbody>
</table>

For the very strong damping forces solution (17) may be used even $x(t, \varepsilon)$ changes rapidly. When $c_1 = 10.1$ or $\lambda = 100\mu = 10$ and $\varepsilon = 0.1$, $x(t, \varepsilon)$ has been calculated by (17) with initial conditions $[x(0) = 1, \dot{x}(0) = -15]$, and correspond numerical solution has been computed. Then percentage errors are calculated and the results are given in Table 3. In this case, the perturbation results almost coincide with numerical results.
For $\varepsilon = 1$, $x(t, \varepsilon)$ has been calculated by (17) in which $a$, $b$ and $u_1$ are evaluated respectively by (23) and (21) with initial conditions $[x(0) = 1, \dot{x}(0) = 0]$ when $c_1 = 4.25$ or $\lambda = 16\mu = 4$. Then corresponding numerical solution has been computed and percentage errors are calculated. The results are given in Table 4. From Table 4, it is clear that most of the times percentage errors are less than 1%.
6. Conclusion

In presence of large or very large linear damping forces, approximate solutions of an over-damped nonlinear system have been found based on the KBM method. The solutions for different initial conditions and as well as for different values of \( \varepsilon \) show good coincidence with corresponding numerical solutions.

Acknowledgment

Author is grateful to referee for his valuable comments and suggestions.

References


Department of Mathematics, Bangladesh Institute of Technology, Rajshahi - 6204, Bangladesh.