ON GAMMA-DERIVATIONS IN GAMMA-NEAR-RINGS

BY

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Abstract. The notion of Γ-derivations in Γ-near-rings is introduced, and then some related properties are discussed.

1. Introduction

As a generalization of near-rings, Γ-near-rings were introduced by Satyanarayana [8]. Booth (together with Groenewald) have studied several aspects in Γ-near-rings (see [1, 2, 3, 4]). The first author together with Sapançi, Öztürk, and Kim have discussed symmetric bi-derivations on prime rings, have dealt with some conditions for traces to be orthogonal on semi-prime Γ-rings, and have studied the extended centroid of prime Γ-rings (see [5, 6, 7]). In this paper, we introduce the notion of Γ-derivations in Γ-near-rings, and investigate basic properties. We give a condition for an additive endomorphism to be a Γ-derivation. We show that a Γ-derivation possessing some conditions must be zero.

2. Preliminaries

All near-rings considered in this paper are left distributive. A Γ-near-ring is a triple \((M, +, \Gamma)\), where

(i) \(\Gamma\) is a nonempty set of binary operators such that \((M, +, \gamma)\) is a near-ring for each \(\gamma \in \Gamma\),

(ii) \(x\gamma(y\mu z) = (x\gamma y)\mu z\) for all \(x, y, z \in M\), and \(\gamma, \mu \in \Gamma\).
For a Γ-near-ring $M$, the set

$$M_0 := \{ x \in M \mid 0\gamma x = 0 \text{ for all } \gamma \in \Gamma \},$$

is called the zerosymmetric part of $M$. A Γ-near-ring $M$ is said to be zerosymmetric if $M = M_0$.

A subset $U$ of a Γ-near-ring $M$ is said to be left (resp. right) invariant if $x\gamma a \in U$ (resp. $a\gamma x \in U$) for all $a \in U$, $\gamma \in \Gamma$ and $x \in M$. If $U$ is both left and right invariant, we say that $U$ is invariant.

If $M$ and $M'$ are Γ-near-rings, then a mapping $f : M \to M'$ such that $f(x + y) = f(x) + f(y)$ and $f(x\gamma y) = f(x)\gamma f(y)$ for all $x, y \in M$ and $\gamma \in \Gamma$ is called a Γ-near-ring homomorphism.

3. Γ-Derivations

In what follows, let $M$ denote a Γ-near-ring unless otherwise specified.

**Definition 3.1.** A Γ-derivation on $M$ is defined to be an additive endomorphism $d$ of $M$ satisfying the product rule $d(x\gamma y) = x\gamma d(y) + d(x)\gamma y$ for all $x, y \in M$ and $\gamma \in \Gamma$.

**Proposition 3.2.** If $d$ is a Γ-derivation on $M$, then

$$d(x\gamma y) = d(x)\gamma y + x\gamma d(y), \quad (3.1)$$

for all $x, y \in M$ and $\gamma \in \Gamma$.

**Proof.** Assume that $d$ is a Γ-derivation on $M$ and let $x, y \in M$ and $\gamma \in \Gamma$. Then

$$d(x\gamma(y + y)) = x\gamma d(y + y) + d(x)\gamma(y + y)$$

$$= x\gamma(d(y) + d(y)) + d(x)\gamma y + d(x)\gamma y$$

$$= x\gamma d(y) + x\gamma d(y) + d(x)\gamma y + d(x)\gamma y. \quad (3.2)$$

On the other hand,

$$d(x\gamma y + x\gamma y) = d(x\gamma y) + d(x\gamma y)$$

$$= x\gamma d(y) + d(x)\gamma y + x\gamma d(y) + d(x)\gamma y. \quad (3.3)$$
Since \(d(x\gamma(y + y)) = d(x\gamma y + x\gamma y)\), it follows that
\[
d(x\gamma y) = x\gamma d(y) + d(x)\gamma y = d(x)\gamma y + x\gamma d(y).
\]
(3.4)

This completes the proof.

**Proposition 3.3.** Every additive endomorphism \(d\) of \(M\) satisfying
\[
d(x\gamma y) = d(x)\gamma y + x\gamma d(y) \quad \text{for all } x, y \in M \text{ and } \gamma \in \Gamma,
\]
is a \(\Gamma\)-derivation on \(M\).

**Proof.** Let \(d\) be an additive endomorphism of \(M\) such that \(d(x\gamma y) = d(x)\gamma y + x\gamma d(y)\) for all \(x, y \in M\) and \(\gamma \in \Gamma\). Then
\[
d(x\gamma(y + y)) = d(x)\gamma(y + y) + x\gamma d(y + y)
= d(x)\gamma y + d(x)\gamma y + x\gamma (d(y) + d(y))
= d(x)\gamma y + d(x)\gamma y + x\gamma d(y) + x\gamma d(y),
\]
(3.5)

and
\[
d(x\gamma y + x\gamma y) = d(x\gamma y) + d(x\gamma y)
= d(x)\gamma y + x\gamma d(y) + d(x)\gamma y + x\gamma d(y).
\]
(3.6)

Since \(x\gamma(y + y) = x\gamma y + x\gamma y\), by comparing (3.5) and (3.6) we have
\[
d(x\gamma y) = d(x)\gamma y + x\gamma d(y) = x\gamma d(y) + d(x)\gamma y.
\]
(3.7)

Hence \(d\) is a \(\Gamma\)-derivation on \(M\).

**Proposition 3.4.** For any \(\Gamma\)-derivation \(d\) on \(M\), we have

(i) \((x\gamma d(y) + d(x)\gamma y)\mu z = x\gamma d(y)\mu z + d(x)\gamma y\mu z,\)

(ii) \((d(x)\gamma y + x\gamma d(y))\mu z = d(x)\gamma y\mu z + x\gamma d(y)\mu z,\)

for all \(x, y, z \in M\) and \(\gamma, \mu \in \Gamma\).

**Proof.** Let \(d\) be a \(\Gamma\)-derivation on \(M\) and let \(x, y, z \in M\) and \(\gamma, \mu \in \Gamma\). Then
\[
d((x\gamma y)\mu z) = (x\gamma y)\mu d(z) + d(x\gamma y)\mu z
= x\gamma y\mu d(z) + (x\gamma d(y) + d(x)\gamma y)\mu z,
\]
(3.8)

and
\[ d(x\gamma(y\mu z)) = x\gamma d(y\mu z) + d(x)\gamma y\mu z \]
\[ = x\gamma(y\mu d(z) + d(y)\mu z) + d(x)\gamma y\mu z \]
\[ = x\gamma y\mu d(z) + x\gamma d(y)\mu z + d(x)\gamma y\mu z. \quad (3.9) \]

Since \( d((x\gamma y)\mu z) = d(x\gamma(y\mu z)) \), it follows that
\[ (x\gamma d(y) + d(x)\gamma y)\mu z = x\gamma d(y)\mu z + d(x)\gamma y\mu z, \quad (3.10) \]
which proves (i). Now using Proposition 3.2, we get
\[ d((x\gamma y)\mu z) = d(x\gamma y)\mu z + x\gamma y\mu d(z) \]
\[ = (d(x)\gamma y + x\gamma d(y))\mu z + x\gamma y\mu d(z), \quad (3.11) \]
and
\[ d(x\gamma(y\mu z)) = d(x)\gamma y\mu z + x\gamma d(y\mu z) \]
\[ = d(x)\gamma y\mu z + x\gamma(d(y)\mu z + y\mu d(z)) \]
\[ = d(x)\gamma y\mu z + x\gamma d(y)\mu z + x\gamma y\mu d(z). \quad (3.12) \]

It follows from \( d((x\gamma y)\mu z) = d(x\gamma(y\mu z)) \) that
\[ (d(x)\gamma y + x\gamma d(y))\mu z = d(x)\gamma y\mu z + x\gamma d(y)\mu z. \quad (3.13) \]

This completes the proof.

A \( \Gamma \)-near-ring \( M \) is said to be prime if \( x\Gamma M\Gamma y = \{0\} \) implies \( x = 0 \) or \( y = 0 \) for all \( x, y \in M \).

**Proposition 3.5.** Let \( M \) be a prime \( \Gamma \)-near-ring and let \( U(\neq \{0\}) \) be a right (resp. left) invariant subset of \( M \). If \( x \) is an element of \( M \) such that \( U\Gamma x = \{0\} \) (resp. \( x\Gamma U = \{0\} \)), then \( x = 0 \).

**Proof.** Assume that \( U(\neq \{0\}) \) is a right invariant subset of \( M \) and let \( x \in M \) be such that \( U\Gamma x = \{0\} \). Taking \( 0 \neq a \in U \), we have \( a\Gamma M\Gamma x \subseteq U\Gamma x = \{0\} \) and so \( a\Gamma M\Gamma x = \{0\} \). Since \( M \) is prime and \( a \neq 0 \), it follows that \( x = 0 \). Similarly for the left case, we get the desired result.

**Proposition 3.6.** Let \( M \) be prime and let \( U(\neq \{0\}) \) be an invariant subset of \( M \). If \( d \) is a nonzero \( \Gamma \)-derivation on \( M \), then for any \( x, y \in M \).
(i) \( x \Gamma U \Gamma y = \{0\} \) implies \( x = 0 \) or \( y = 0 \).

(ii) \( d(U) \Gamma y = \{0\} \) implies \( y = \{0\} \).

(iii) \( M \) is zerosymmetric and \( x \Gamma d(U) = \{0\} \) imply \( x = 0 \).

**Proof.** (i) Let \( x, y \in M \) be such that \( x \Gamma U \Gamma y = \{0\} \). Then \( x \Gamma U \Gamma y \subseteq x \Gamma U \Gamma y = \{0\} \). Since \( M \) is prime, it follows that \( x \Gamma U = \{0\} \) or \( y = 0 \) so from Proposition 3.5 that \( x = 0 \) or \( y = 0 \).

(ii) Assume that \( d(U) \Gamma y = \{0\} \) for \( y \in M \) and let \( a \in U, x \in M, \) and \( \gamma, \mu \in \Gamma \). Then

\[
0 = d(x \gamma a \mu y)
= (x \gamma d(a) + d(x \gamma a) \mu y) \quad \text{[because } d \text{ is a } \Gamma \text{-derivation]}
= x \gamma d(a) \mu y + d(x \gamma a) \mu y \quad \text{[by Proposition 3.4(i)]}
= d(x \gamma a) \mu y, \quad \text{[because } d(U) \Gamma y = \{0\}] \tag{3.14}
\]

and so \( d(x) \Gamma U \Gamma y = \{0\} \). Since \( d \) is nonzero, it follows from (i) that \( y = 0 \).

(iii) Suppose that \( M \) is zerosymmetric and \( x \Gamma d(U) = \{0\} \) for \( x \in M \) and let \( a \in U, y \in M, \) and \( \gamma, \mu \in \Gamma \). Then

\[
0 = x \gamma d(a \mu y)
= x \gamma (a \mu d(y) + d(a) \mu y)
= x \gamma a \mu d(y) + x \gamma d(a) \mu y
= x \gamma a \mu d(y), \quad \text{[by Proposition 3.4(ii)]} \tag{3.15}
\]

which implies that \( x \Gamma U \Gamma d(y) = \{0\} \). Since \( d \) is nonzero, it follows from (i) that \( x = 0 \). This completes the proof.

**Proposition 3.7.** Let \( M \) be zerosymmetric and prime and let \( U(\neq \{0\}) \) be an invariant subset of \( M \). If \( d \) is a nonzero \( \Gamma \)-derivation on \( M \) such that \( d^2(U) = 0 \), then \( d^2 = 0 \).

**Proof.** Note that \( d^2(u \gamma v) = 0 \) for all \( u, v \in U \) and \( \gamma \in \Gamma \). Thus

\[
0 = d^2(u \gamma v) = d(d(u \gamma v))
= d(u \gamma d(v) + d(u) \gamma v)
\]
\[
= d(u \gamma d(v)) + d(d(u) \gamma v) \\
= u \gamma d^2(v) + d(u) \gamma d(v) + d(u) \gamma d(v) + d^2(u) \gamma v \\
= u \gamma d^2(v) + 2d(u) \gamma d(v) + d^2(u) \gamma v \\
= 2d(u) \gamma d(v),
\]

and so \(2d(u) \Gamma \Gamma d(U) = \{0\} \) for all \(u \in U\). It follows from Proposition 3.6 that \(2d(u) = 0\). Now for all \(y \in M, \mu \in \Gamma \) and \(u \in U\), we have

\[
d^2(\mu \gamma y) = d(d(\mu \gamma y)) = d(\mu \gamma d(y) + d(u) \mu y) \\
= d(\mu \gamma d(y)) + d'(d(u) \mu y) \\
= u \mu d^2(y) + d(u) \mu d(y) + d(u) \mu d(y) + d^2(u) \mu y \\
= u \mu d^2(y) + 2d(u) \mu d(y) + d^2(u) \mu y,
\]

which implies that \(u \mu d^2(y) = 0\), that is, \(U \Gamma d^2(y) = \{0\} \) for all \(y \in M\). Applying Proposition 3.5 yields \(d^2(y) = 0\) for all \(y \in M\), that is, \(d^2 = 0\). This completes the proof.

**Proposition 3.8.** Let \(M\) be zerosymmetric and prime and let \(x(\neq 0) \in M\). Then every \(\Gamma\)-derivation \(d\) on \(M\) satisfying \(x \gamma d(y) = 0\) for all \(y \in M\) and \(\gamma \in \Gamma\) is zero.

**Proof.** Let \(x(\neq 0) \in M\) and let \(d\) be a \(\Gamma\)-derivation on \(M\) such that \(x \gamma d(y) = 0\) for all \(y \in M\) and \(\gamma \in \Gamma\). Taking \(y \mu z\) instead of \(y\) yields

\[
0 = x \gamma d(y \mu z) = x \gamma (y \mu d(z) + d(y) \mu z) \\
= x \gamma y \mu d(z) + x \gamma d(y) \mu z = x \gamma y \mu d(z),
\]

which implies that \(x \gamma y \mu d(z) = 0\) for all \(z \in M\). Since \(M\) is prime and \(x \neq 0\), it follows that \(d(z) = 0\) for all \(z \in M\). Hence \(d\) is zero.

**Theorem 3.9.** Let \(M\) be prime and 2-torsion free and let \(d_1\) and \(d_2\) be \(\Gamma\)-derivations on \(M\) such that

(i) \(d_1 d_2\) is also a \(\Gamma\)-derivation on \(M\),
(ii) \(d_1(x) \gamma d_2(y) = d_2(y) \gamma d_1(x)\) for all \(x, y \in M\) and \(\gamma \in \Gamma\).
Then either \( d_1 = 0 \) or \( d_2 = 0 \).

**Proof.** Let \( d = d_1 d_2 \) and let \( x, y \in M \) and \( \gamma \in \Gamma \). Then

\[
d(x\gamma y) = x\gamma d(y) + d(x)\gamma y.
\] (3.19)

On the other hand,

\[
d(x\gamma y) = d_1(d_2(x\gamma y))
\]
\[
= d_1(x\gamma d_2(y) + d_2(x)\gamma y)
\]
\[
= d_1(x\gamma d_2(y)) + d_1(d_2(x)\gamma y)
\]
\[
= x\gamma d(y) + d_1(x)\gamma d_2(y) + d_2(x)\gamma d_1(y) + d(x)\gamma y.
\] (3.20)

Combining (3.19) and (3.20) yields

\[
d_1(x)\gamma d_2(y) + d_2(x)\gamma d_1(y) = 0.
\] (3.21)

Taking \( x\mu d_2(z) \) instead of \( x \) in (3.21) where \( \mu \in \Gamma \), we get

\[
0 = d_1(x\mu d_2(z))\gamma d_2(y) + d_2(x\mu d_2(z))\gamma d_1(y)
\]
\[
= (d_1(x)\mu d_2(z) + x\mu d(z))\gamma d_2(y) + (x\mu d_2^2(z) + d_2(x)\mu d_2(z))\gamma d_1(y)
\]
\[
= d_1(x)\mu d_2(z)\gamma d_2(y) + x\mu d(z)\gamma d_2(y) + x\mu d_2(z)\gamma d_2(y) + d_2(x)\mu d_2(z)\gamma d_1(y)
\]
\[
= d_1(x)\mu d_2(z)\gamma d_2(y) + x\mu (d(z)\gamma d_2(y) + d_2^2(z)\gamma d_1(y)) + d_2(x)\mu d_2(z)\gamma d_1(y).
\] (3.22)

Now, taking \( d_2(z) \) instead of \( x \) in (3.21) we have

\[
0 = d_1 d_2(z)\gamma d_2(y) + d_2 d_2(z)\gamma d_1(y)
\]
\[
= d(z)\gamma d_2(y) + d_2^2(z)\gamma d_1(y),
\] (3.23)

and hence \( x\mu (d(z)\gamma d_2(y) + d_2^2(z)\gamma d_1(y)) = 0 \). It follows from (3.22) that

\[
d_1(x)\mu d_2(z)\gamma d_2(y) + d_2(x)\mu d_2(z)\gamma d_1(y) = 0.
\] (3.24)

Replacing \( x \) by \( z \) in (3.21), we have

\[
d_2(z)\gamma d_1(y) = -d_1(z)\gamma d_2(y).
\] (3.25)

Taking \( z \) and \( \mu \) instead of \( y \) and \( \gamma \) in (3.21) respectively, we get

\[
d_1(x)\mu d_2(z) = -d_2(x)\mu d_1(z).
\] (3.26)
Combining (3.24), (3.25) and (3.26), we obtain
\[ 0 = -d_2(x)\mu d_1(z)\gamma d_2(y) + d_2(x)\mu(-d_1(z)\gamma d_2(y)) \]
\[ = d_2(x)\mu((-d_1(z))\gamma d_2(y) + (-d_1(z))\gamma d_2(y)) \]
\[ = d_2(x)\mu(2(-d_1(z))\gamma d_2(y)). \]

(3.27)

Assume that \( d_2 \neq 0 \). Since \( M \) is prime, it follows that \( 2(-d_1(z))\gamma d_2(y) = 0 \) so that \( d_1(z)\gamma d_2(y) = 0 \) because \( M \) is 2-torsion free. This implies that \( d_1(z) = 0 \) for all \( z \in M \), i.e., \( d_1 = 0 \). This completes the proof.

**Acknowledgment**

The first author was supported by Korea Research Foundation Grant (KRF-2001-005-D00002).

**References**


