VECTOR VALUED PARANORMED BOUNDED AND NULL SEQUENCE SPACES ASSOCIATED WITH MULTIPLIER SEQUENCES

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Abstract. In this article we introduce the multiplier vector valued sequence spaces $\ell_\infty\{E_k, \Lambda, p\}$ and $c_0\{E_k, \Lambda, p\}$, where $\Lambda = (\gamma_k)$ is an associated multiplier sequence of non-zero complex numbers and the terms of the sequence are chosen from the seminormed spaces $E_k$, $k \in N$. This generalizes the scalar sequence spaces $\ell_\infty\{p\}$ and $c_0\{p\}$. We study some properties of these spaces like solidity, completeness and prove some inclusion results. We characterize the multiplier problem and obtain their duals.

1. Introduction

Let $w$ denote the set of all sequences of complex terms. Then $w$ is a linear space under co-ordinatewise addition and scalar multiplication. Any subspace of $w$ is called as a sequence space. For example $\ell_\infty$, the set of all bounded complex sequences is a sequence space.

The notion of paranormed sequence space was introduced by Nakano [10] and Simons [15]. It was further investigated from sequence space point of view and linked with summability theory by Maddox [8, 9], Lascarides [6], Nanda [11], Ratha [13], Rath and Tripathy [12] and many others.

The studies on vector valued sequence spaces is done by Gupta [3], Ratha and Srivastava [14], Das and Choudhury [1], Leonard [7], Tripathy and Sarma [16] and many others.

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The scope for the studies on sequence spaces was extended by using the notion of associated multiplier sequences. Goes et al. [2] defined the differentiated sequence space $dE$ and integrated sequence space $\int E$ for a given sequence space $E$, using the multiplier sequences $(k^{-1})$ and $(k)$ respectively. Kamthan [4] used the multiplier sequence $(k!)$. In the present article we shall consider a general multiplier sequence $\Lambda = (\gamma_k)$ of non-zero scalars.

Let $\Lambda = (\gamma_k)$ be a sequence of non-zero scalars. Then for $E$ a sequence space, the multiplier sequence space $E(\Lambda)$, associated with the multiplier sequence $\Lambda$ is defined as

$$E(\Lambda) = \{(x_k) \in w : (\gamma_k x_k) \in E\}.$$  

**Example 1.** Let $\gamma_k = k^{-1}$ for all $k \in N$. Then

$$c_0(k^{-1}) = \{(x_k) \in w : k^{-1}x_k \to 0, \text{ as } k \to \infty\}.$$  

From the above example it is clear that $c_0(k^{-1})$ contains some unbounded sequences too. Further it accelerates the convergence of the sequences in $c_0$. It also covers a larger class of sequences for the study. Some times the associated multiplier sequence delays the rate of convergence of a sequence.

During a chemical reaction, a catalyst is used to accelerate the process of reaction. For example AIBN is used as a catalyst for polymerization.

2. Definitions and Preliminaries

Let $X$ be a linear space and $g : X \to R$ is such that

(i) $g(x) \geq 0$.
(ii) $x = \theta$ implies $g(x) = 0$.
(iii) $g(x + y) \leq g(x) + g(y)$.
(iv) $g(-x) = g(x)$, for all $x \in X$.
(v) $g(\lambda_n x_n - \lambda x) \to 0$, as $n \to \infty$, for scalars $\lambda_n$ and $\lambda$ and $x_n, x \in X$, whenever $\lambda_n \to \lambda$ and $x_n \to x$, as $n \to \infty$.

Then $g$ is said to be a paranorm on $X$ and $(X, g)$ is called a paranormed space.
Example 2. Let \( p = (p_k) \) be such that \( p \in \ell_\infty \) and \( M = \max(1, \sup p_k) \). Then
\[
\ell_\infty(p) = \{ (x_k) \in w : \sup_k |x_k|^{p_k} < \infty \}
\]
is a paranormed space, paranormed by \( g(x_k) = \sup_k |x_k|^{\frac{p_k}{M}} \).

A vector valued sequence space \( E \) is called solid (or normal) if \( \alpha x = (\alpha_k x_k) \in E \), whenever \( x = (x_k) \in E \) and \( \alpha = (\alpha_k) \) is a sequence of scalars such that \(|\alpha_k| \leq 1\) for all \( k \in \mathbb{N} \). A sequence space \( E \) is said to be monotone if \( E \) contains the canonical preimages of all its subspaces (one may refer to Kamthan and Gupta [5], p.48).

Throughout the article \( E_k \) will denote a seminormed sequence space, semi-normed by \( f_k \) for all \( k \in \mathbb{N} \), defined over \( C \), the field of complex numbers. Throughout \( p = (p_k) \) is a sequence of strictly positive numbers and \( t_k = \frac{1}{p_k} \), for all \( k \in \mathbb{N} \).

We define the following vector valued multiplier sequence spaces:
\[
\ell_\infty(E_k, \Lambda, p) = \{ (x_k) : x_k \in E_k \text{ for all } k \in \mathbb{N} \text{ and } \sup_k (f_k(\gamma_k x_k))^{p_k} < \infty \},
\]
\[
c_0(E_k, \Lambda, p) = \{ (x_k) : x_k \in E_k \text{ for all } k \in \mathbb{N} \text{ and } (f_k(\gamma_k x_k))^{p_k} \to 0, \text{ as } k \to \infty \},
\]
\[
\ell_\infty(E_k, \gamma, p) = \{ (x_k) : x_k \in E_k \text{ for all } k \in \mathbb{N} \text{ and there exists } r > 0 \text{ such that } \sup_k (f_k(r\gamma_k x_k))^{p_k} t_k < \infty \},
\]
\[
c_0(E_k, \Lambda, p) = \{ (x_k) \in E_k \text{ for all } k \in \mathbb{N} \text{ and there exists } r > 0 \text{ such that } (f_k(r\gamma_k x_k))^{p_k} t_k \to 0, \text{ as } k \to \infty \}.
\]

Two sequence spaces \( E \) and \( F \) are said to be equivalent if there exists a sequence \( u = (u_k) \) of strictly positive numbers such that the mapping
\[
u : E \to F \text{ defined by } y = ux = (u_k x_k) \in F, \text{ whenever } (x_k) \in E,
\]
is one-to-one correspondence between \( E \) and \( F \). It is denoted by \( E \cong F(u) \) or simply \( E \cong F \) (see for instance Nakano [10]).

It is remarked by Lascarides [6] (Remark 3) that “If \( E \) is a sequence space paranormed (or normed) by \( g \) and \( E \cong F(u) \), then \( F \) is a sequence space paranormed (or normed) by \( g_u \) defined by \( g_u(y) = g(u^{-1}y), y \in F \).
Further it is noted by Lascarides [6] that “if \((p_k) \in \ell_\infty\), then \(c_0(p) \cong c_0\{u\}\), (as well as \(\ell_\infty(p) \cong \ell_\infty\{p\}\{u\}\), where \(u = (p_k^t)\)”.

For \(E, F\) two sequence spaces we define \(M(F, E)\) as follows:

\[
M(F, E) = \{ (\gamma_k) : (\gamma_k x_k) \in E \text{ for all } (x_k) \in F \},
\]

where \(\Lambda = (\gamma_k)\) is a multiplier sequence.

**Lemma 1.** A sequence space \(E\) is solid implies \(E\) is monotone.

**Lemma 2.** (Lascarides [6], Proposition 1) Let \(h = \inf p_k\) and \(H = \sup p_k\).

Then the following conditions are equivalent:

(i) \(H < \infty\) and \(h > 0\).

(ii) \(c_0(p) = c_0\) or \(\ell_\infty(p) = \ell_\infty\).

(iii) \(\ell_\infty\{p\} = \ell_\infty(p)\).

(iv) \(c_0\{p\} = c_0(p)\).

(v) \(\ell\{p\} = \ell(p)\).

3. Main Results

In this section we prove the results involving \(\ell_\infty\{E_k, \Lambda, p\}\) and \(c_0\{E_k, \Lambda, p\}\).

First we prove the following properties of the spaces \(\ell_\infty\{E_k, \Lambda, p\}\) and \(c_0\{E_k, \Lambda, p\}\).

**Property 1.** \(c_0\{E_k, \Lambda, p\}\) is a linear space for any sequence \(p = (p_k)\).

**Proof.** Let \(x \in c_0\{E_k, \Lambda, p\}\). Then there exists \(r > 0\) such that \((f_k(r \gamma_k x_k))^{p_k} t_k \to 0\), as \(k \to \infty\). Let \(\lambda \in C\). Without loss of generality we can take \(\lambda \neq 0\). Let \(\rho = r(|\lambda|)^{-1} > 0\), then we have

\[
(f_k(\gamma_k(\lambda x_k)\rho))^{p_k} t_k = (f_k(r \gamma_k x_k))^{p_k} t_k \to 0, \quad \text{as } k \to \infty.
\]

Therefore \(\lambda x \in c_0\{E_k, \Lambda, p\}\), for all \(\lambda \in C\) and for all \(x \in c_0\{E_k, \Lambda, p\}\).

Next we suppose that \(x, y \in c_0\{E_k, \Lambda, p\}\). Then there exist \(r_1, r_2 > 0\) such that

\[
(f_k(\gamma_k x_k r_1))^{p_k} t_k \to 0, \quad \text{as } k \to \infty.
\]
and
\[(f_k(\gamma_kykr_2))^{pk}t_k \to 0, \text{ as } k \to \infty.\]

Thus given \(\varepsilon > 0\), there exists \(k_1, k_2 > 0\) such that
\[(f_k(\gamma_kxkr_1))^{pk}t_k < \varepsilon p_k, \text{ for all } k \geq k_1\] (1)
and
\[(f_k(\gamma_kykr_2))^{pk}t_k < \varepsilon p_k, \text{ for all } k \geq k_2.\] (2)

Let \(r = r_1r_2(r_1 + r_2)^{-1}\) and \(k_0 = \max(k_1, k_2)\).

Then we have for all \(k \geq k_0\),
\[[f_k(\gamma_k(x_k + y_k)r)]^{pk} \leq [f_k(\gamma_kxkr_1)r_2(r_1 + r_2)^{-1} + f_k(\gamma_kykr_2)r_1(r_1 + r_2)^{-1}]^{pk} < \varepsilon p_k.\]

Hence \(x + y \in c_0\{E_k, \Lambda, p\}\).

Thus \(c_0\{E_k, \Lambda, p\}\) is a linear space.

In view of the techniques applied in proving Property 1, the proof of the following is a routine work.

**Property 2.** \(\ell_\infty\{E_k, \Lambda, p\}\) is linear for any sequence \(p = (p_k)\).

**Theorem 1.** If \(p \in \ell_\infty\) and each \(E_k\) is complete, then \(c_0\{E_k, \Lambda, p\}\) is a complete paranormed space, paranormed by
\[g(x) = \sup_k(f_k(\gamma_kx_kp_k^{-t_k})^{pk})^{\frac{1}{M}},\]
where \(M = \max(1, H), H = \sup_k p_k\).

**Proof.** Clearly, for any \(x \in c_0\{E_k, \Lambda, p\}\)
\[g(x) \geq 0, \quad g(\theta) = 0 \quad \text{and} \quad g(-x) = g(x).\]

Now, let \(x, y \in c_0\{E_k, \Lambda, p\}\). Then clearly \(g(x + y) \leq g(x) + g(y)\).

Now we check the continuity of scalar multiplication. It can be shown by standard techniques that \(\alpha \to 0, x \to \theta\) imply \(\alpha x \to \theta\) and \(\alpha\) fixed, \(x \to \theta\) imply \(\alpha x \to \theta\).

Next we show that \(\alpha \to 0\) and \(x\) fixed imply \(\alpha x \to \theta\).

We have \(g(\alpha x) = \sup_k(f_k(\gamma_k(\alpha x_k)p_k^{-t_k})^{pk})^{\frac{1}{M}}.\)
Now, \((x_k) \in c_0\{E_k, \Lambda, p\}\). Then there exists \(r > 0\) such that
\[
\min(1, r^H)(f_k(\gamma_k x_k))^{pk} t_k \leq (f_k(\gamma_k x_k))^{pk} t_k \to 0 \text{ as } k \to \infty.
\]
(since \(\min(1, r^H) \leq r^{pk}\) for all \(k \in N\))

\[
\Rightarrow (f_k(\gamma_k x_k p_k^{-t_k}))^{pk} t_k \to 0 \text{ as } k \to \infty.
\]

\[
\Rightarrow \text{Given } \epsilon > 0, \text{ there exists } k_0 \in N \text{ such that } (f_k(\gamma_k x_k p_k^{-t_k}))^{pk} t_k < \epsilon, \text{ for all } k > k_0.
\]

Since \(\alpha \to 0\), without loss of generality let \(|\alpha| < 1\), then we have
\[
(f_k(\gamma_k (\alpha x_k)p_k^{-t_k}))^{pk} t_k < \epsilon, \text{ for all } k > k_0.
\]

Again \(x \in c_0\{E_k, \Lambda, p\}\) implies there exists \(L < \infty\) such that
\[
(f_k(\gamma_k x_k p_k^{-t_k}))^{pk} t_k \leq L, \text{ for } k = 1, 2, 3, \ldots, k_0.
\]

For \(k = 1, 2, 3, \ldots, k_0\), let \(|\alpha|^{|\frac{1}{pk}|} < \delta(= \frac{\epsilon}{L})\), then we have
\[
(f_k(\gamma_k ((\alpha x_k)p_k^{-t_k}))^{pk} t_k < \epsilon, \text{ for all } k = 1, 2, 3, \ldots, k_0.
\]

Taking \(\alpha\) small enough, we have
\[
(f_k(\gamma_k (\alpha x_k)p_k^{-t_k}))^{pk} t_k < \epsilon, \text{ for all } k \in N.
\]

So, \(g(\alpha x) \to 0\) as \(\alpha \to 0\).

Hence \(g\) is a paranorm on \(c_0\{E_k, \Lambda, p\}\).

We now show that \(c_0\{E_k, \Lambda, p\}\) is complete under the paranorm \(g\). For this, let \((x^{(n)})\) be a Cauchy sequence in \(c_0\{E_k, \Lambda, p\}\). Then given \(\epsilon > 0\), there exists \(n_0 \in N\) such that
\[
g(x^{(n)} - x^{(m)}) < \epsilon, \text{ for all } n, m \geq n_0
\]
\[
\Rightarrow f_k(\gamma_k x_k^{(n)} p_k^{-t_k} - \gamma_k x_k^{(m)} p_k^{-t_k}) < e^{pk} t_k < \epsilon,
\]
for all \(n, m \geq n_0\) and for all \(k \in N\). \(3\)

We write \(z_k^{(n)} = \gamma_k x_k^{(n)} p_k^{-t_k}\), for all \(k \in N\).

Then \((z_k^{(n)})_{n \in N}\) is a Cauchy sequence in \(E_k\) for each \(k \in N\). Since \(E_k\)'s are complete there exists \(z_k \in E_k\) such that \(z_k^{(n)} \to z_k\) as \(n \to \infty\), for all \(k \in N\).
Since $E_k$’s are linear, we can express $z_k$ as $z_k = \gamma_k x_k p_k^{-t_k}$ where $x_k \in E_k$. Let $x = (x_k)$.

Taking $m \to \infty$ in (3), we get

$$\sup_k (f_k(\gamma_k(x_k^{(n)} - x_k)p_k^{-t_k}))^{\frac{p_k}{t_k}} < \epsilon, \quad \text{for all } n \geq n_0.$$ 

$$\Rightarrow g(x^{(n)} - x) < \epsilon, \quad \text{for all } n \geq n_0.$$ 

$$\Rightarrow x^{(n)} \to x, \quad \text{as } n \to \infty. \quad (4)$$

Also (4) implies that $x^{(n)} - x \in c_0\{E_k, \Lambda, p\}$. Since $c_0\{E_k, \Lambda, p\}$ is linear, we have $x = (x^{(n)} - x) + x^{(n)} \in c_0\{E_k, \Lambda, p\}$.

Hence $c_0\{E_k, \Lambda, p\}$ is complete under the paranorm $g$.

**Corollary 1.** If $0 < \inf p_k \leq \sup p_k < \infty$, then $\ell_\infty\{E_k, \Lambda, p\}$ is paranormed by $g$.

**Proof.** In view of Theorem 1, we need only to check that $\alpha \to 0$ and $x$ fixed imply $\alpha x \to 0$.

We have, $(g(\alpha x))^M \leq \sup_k |\alpha|^{p_k}(g(x))^M$.

Since $\alpha \to 0$, without loss of generality we can take $|\alpha| < 1$. Now if $\inf p_k = 0$, then the right side of the above inequality will be independent of $\alpha$.

Hence the result.

Using Corollary 1 and following Theorem 1 one will get the following result.

**Theorem 2.** If $0 < \inf p_k \leq \sup p_k < \infty$ and each $E_k$ is complete, then $\ell_\infty\{E_k, \Lambda, p\}$ is a complete paranormed space paranormed by $g$.

**Proposition 3.** The space $c_0\{E_k, \Lambda, p\}$ and $\ell_\infty\{E_k, \Lambda, p\}$ are normal.

**Proof.** Let $x$ be an element of either of the spaces and $|\alpha_k| \leq 1$, for $k = 1, 2, 3, \ldots$.

Since $|\alpha_k|^{p_k} \leq \max(1, |\alpha_k|^H) \leq 1$, for all $k \in N$, so

$$(f_k(\gamma_k(\alpha_k x_k)r))^{p_k} t_k \leq (f_k(\gamma_k x_k r))^{p_k} t_k.$$ 

Thus $x \in X$ and $|\alpha_k| \leq 1$ for all $k \in N$ imply $\alpha x \in X$ where $X$ is either $c_0\{E_k, \Lambda, p\}$ or $\ell_\infty\{E_k, \Lambda, p\}$. 
Hence the result.

The next result follows immediately from Lemma 1 and Proposition 3.

**Proposition 4.** The spaces $c_0\{E_k, \Lambda, p\}$ and $\ell_\infty\{E_k, \Lambda, p\}$ are monotone.

**Theorem 5.** Let $(p_k)$ be a given sequence of strictly positive real numbers. Then $(\gamma_k) \in (E,E)$ if and only if $(\gamma_k)^{p_k}) \in \ell_\infty$, where $E = c_0\{E_k, p\}$ or $\ell_\infty\{E_k, p\}$ or $c_0(E_k, p)$ or $\ell_\infty(E_k, p)$.

**Proof.** The sufficiency for all the cases is obvious.

For the necessity, for the case $E = \ell_\infty\{E_k, p\}$, suppose that $(\gamma_k)^{p_k}) \notin \ell_\infty$. Then there exists a subsequence $(\gamma_{k_i})^p$ such that $\gamma_{k_i} \rightarrow \infty$, as $i \rightarrow \infty$.

Then $|(\gamma_{k_i})^{p_{k_i}}| \geq (f_{k_i}(r\gamma_{k_i}x_{k_i}))^{p_k_{k_i}}$, for all $i \in N$.

Hence $(\gamma_k) \notin (\ell_\infty\{E_k, p\}, \ell_\infty\{E_k, p\})$.

Next for $E = c_0\{E_k, p\}$, for the necessity, suppose that (5) holds. Then one can find a sequence $(x_k) \in c_0\{E_k, p\}$ such that

Then $|(\gamma_{k_i})^{p_{k_i}}| \geq (f_{k_i}(r\gamma_{k_i}x_{k_i}))^{p_k_{k_i}}$, for all $i \in N$.

Thus $(\gamma_k) \notin (c_0\{E_k, p\}, c_0\{E_k, p\})$. As such we arrive at a contradiction.

Hence $(\gamma_k)^{p_k}) \in \ell_\infty$ is necessary for $(\gamma_k) \in (\ell_\infty\{E_k, p\}, \ell_\infty\{E_k, p\})$.

The other cases can be proved similarly.

The following result follows from Lemma 2 and Theorem 5.

**Corollary 2.** $M(E, E) = \ell_\infty$, for $E = c_0\{E_k, p\}$ or $\ell_\infty\{E_k, p\}$ or $c_0(E_k, p)$ or $\ell_\infty(E_k, p)$ if and only if $h > 0$ and $H < \infty$.

From Lemma 2 and Corollary 2, the following result follows.

**Corollary 3.** Let $h = \inf p_k$ and $H = \sup p_k$. Then the following are equivalent:
(i) $H < \infty$ and $h > 0$.
(ii) $\ell_\infty\{E_k, \Lambda, p\} = \ell_\infty(E_k, \Lambda, p)$.
(iii) $c_0\{E_k, \Lambda, p\} = c_0(E_k, \Lambda, p)$.

**Theorem 6.** If $p, q \in \ell_\infty$. Then $\ell_\infty\{E_k, \Lambda, p\} \subseteq \ell_\infty\{E_k, \Lambda, q\}$ if and only if

$$\liminf_k q_k (M p_k)^{-\frac{q_k}{p_k}} > 0$$

(6)

for every integer $M > 1$.

**Proof. Sufficiency.** Let $x \in \ell_\infty\{E_k, \Lambda, p\}$. Then there exists $r > 0$ such that $\sup k f_k(\gamma_k x_k r)^{p_k} t_k < \infty$. So there exists $M = M(r) > 1$ such that

$$f_k(\gamma_k x_k r)^{p_k} \leq M p_k \text{ for all } k \in \mathbb{N}. \quad (7)$$

Also (6) implies there exists $\alpha > 0$ such that

$$q_k (M p_k)^{-\frac{q_k}{p_k}} > \alpha$$

(8)

for all sufficiently large $k$.

Now, $(f_k(\gamma_k x_k r))^{q_k} q_k^{-1} \leq (M p_k)^{\frac{q_k}{p_k}} q_k^{-1}$ [by (7)]

$$< \alpha^{-1}, \text{ for sufficiently large } k \text{ [by (8)].}$$

So, $(x_k) \in \ell_\infty\{E_k, \Lambda, q\}$.
Hence $\ell_\infty\{E_k, \Lambda, p\} \subseteq \ell_\infty\{E_k, \Lambda, q\}$.

**Necessity.** Suppose $\ell_\infty\{E_k, \Lambda, p\} \subseteq \ell_\infty\{E_k, \Lambda, q\}$ but there exists $M_0 > 1$ such that

$$\liminf_k q_k (M_0 p_k)^{-\frac{q_k}{p_k}} = 0.$$ 

Then there exists $k_1 < k_2 < k_3 < \cdots$ such that

$$q_{k_i} (M_0 p_{k_i})^{-\frac{q_{k_i}}{p_{k_i}}} < \frac{1}{i-1}, i = 1, 2, 3, \ldots.$$ 

Let $H' = \sup_k q_k < \infty$. Define the sequence $x = (x_k)$ as follows:

$$x_k = \begin{cases} \frac{(M_0 p_k)^\frac{q_k}{p_k}}{q_k} I_k, & k = k_i, \\ \theta_k, & \text{otherwise,} \end{cases}$$
where $\theta_k$ is the zero element and $I_k$ the identity of $E_k$ for all $k \in N$.

Then $(f_k(\gamma_k x_k 1))^{p_k=p_k-1} = M_0$, for every $k = k_i$, $i = 1, 2, 3, \ldots$.

Now $(x_k) \in \ell_\infty\{E_k, \Lambda, p\}$, but

$$(f_k(\gamma_k x_k r))^{q_k} q_k^{-1} = (M_0 p_k)^{q_k} r^{p_k} q_k^{-1}, \quad \text{for every } k = k_i, i = 1, 2, 3, \ldots$$

and for every $r > 0$,

$$> i \min(1, r^{H'}) .$$

Therefore $(x_k) \notin \ell_\infty\{E_k, \Lambda, q\}$. Hence we arrive at a contradiction.

Thus (6) holds.

**Theorem 7.** Let $p, q \in \ell_\infty$. Then $c_0\{E_k, \Lambda, p\} \subseteq c_0\{E_k, \Lambda, q\}$ if and only if

$$\lim_{M} \limsup_{k} q_k^{-1}(M^{-1} p_k)^{q_k} = 0 .$$

(9)

**Proof.** Let $I(M) = \limsup_{k} q_k^{-1}(M^{-1} p_k)^{q_k}$, for all $M > 1$ and $I(M, k) = q_k^{-1}(M^{-1} p_k)^{q_k}$.

By (9) given $\epsilon > 0$, there exists $M_0 > 1$ such that

$$I(M) < \epsilon, \text{ for all } M > M_0 .$$

(10)

Let $x \in c_0\{E_k, \Lambda, p\}$ and $M^*$ be fixed with $M^* > M_0$. Then there exists $r > 0$ and $k_0 \in N$ such that

$$(f_k(\gamma_k x_k r))^{p_k=p_k-1} < M^{*-1}, \text{ for all } k > k_0 .$$

Now, $(f_k(\gamma_k x_k r))^{q_k} q_k^{-1} = q_k^{-1}(M^{*-1} p_k)^{q_k} < I(M^*) < \epsilon, \text{ for all } k > k_0$. [by (10)]

So, $x \in c_0\{E_k, \Lambda, p\}$.

Hence $c_0\{E_k, \Lambda, p\} \subseteq c_0\{E_k, \Lambda, q\}$.

Conversely suppose that $c_0\{E_k, \Lambda, p\} \subseteq c_0\{E_k, \Lambda, q\}$ but (9) does not hold.

Then we have two cases:

(i) $I(M) = \infty$ for every integer $M > 1$.

(ii) There exists $M_0 > 1$ such that $I(M) < \epsilon$, for all $M > M_0$ and $\lim_M I(M) > 0$. 

Case (i) Consider a strictly increasing sequence \((k_i)\) of positive integers such that
\[ I(i + 1, k_i) > i, \quad i = 1, 2, 3, \ldots \]
Define \((x_k)\) as follows:
\[
x_k = \begin{cases} 
\frac{(i+1)^{i^k}p_kI_k}{|\gamma_k|}, & k = k_i, \\
\theta_k, & \text{otherwise.}
\end{cases}
\]
So, \((f_k(\gamma_kx_k1))^{p_k}p_k^{-1} = (i + 1)^{-1}, \) for all \(k = k_i, \ i = 1, 2, 3, \ldots\)
Hence \((x_k) \in c_0\{E_k, \Lambda, p\}\).
Also, \((f_k(\gamma_kx_kr))^{q_k}q_k^{-1} = r^{q_k}q_k^{-1}p_kr_k(i + 1)^{-\frac{q_k}{p_k}} \geq I(i + 1, k_i) \min(1, r^{H'})\), for \(k = k_i\) and \(r > 0\)
\[
(H' = \sup_k q_k < \infty) > i \min(1, r^{H'}).
\]
Therefore \((x_k) \notin c_0\{E_k, \Lambda, q\}\). Hence we arrive at a contradiction.

Case (ii) Suppose \(\lim_{M > M_0} I(M) = 2a > 0\). Therefore there exists a strictly increasing sequence \((k_i)\) of positive integers such that
\[ I(M_0 + i - 1, k_i) > a, \quad i = 1, 2, 3, \ldots \]
We define a sequence \(x = (x_k)\) as follows:
\[
x_k = \begin{cases} 
\frac{(M_0+i-1)^{i^k}p_kI_k}{|\gamma_k|}, & k = k_i, \\
\theta_k, & \text{otherwise.}
\end{cases}
\]
Then \((f_k(\gamma_kx_k1))^{p_k}p_k^{-1} = (M_0 + i - 1)^{-1}, \) for all \(k = k_i, i = 1, 2, 3, \ldots\)
So, \((x_k) \in c_0\{E_k, \Lambda, p\}\).
Also for \(k = k_i, i = 1, 2, 3, \ldots\), we have
\[
(f_k(\gamma_kx_kr))^{q_k}q_k^{-1} = r^{q_k}q_k^{-1}p_kr_k(M_0 + i - 1)^{-\frac{q_k}{p_k}} \geq I(M_0 + i - 1, k_i) \min(1, r^{H'}) > a \min(1, r^{H'}). 
\]
Thus \((x_k) \notin c_0\{E_k, \Lambda, q\}\). Once again we arrive at a contradiction. Therefore \(\lim_M I(M) = 0\).

This completes the proof of the theorem.

Remark. Taking \(E_k\)'s as normed linear spaces, normed by \(\|\cdot\|_{E_k}\), one will get the analogue of the above results for \(\ell_\infty\{E_k, \Lambda, p\}\) and \(c_0\{E_k, \Lambda, p\}\). These spaces are paranormed by

\[ f(x) = \sup_k \{\|\gamma_k x_k\|_{E_k} p_{-t_k}^{\frac{p_k}{p}}\}. \]

4. Duals of the Above Sequence Spaces

By \(\ell_1\) we denote the class of \textit{absolutely summable} sequence spaces of complex terms. The \(\alpha\)-dual i.e. the Köthe-Toeplitz dual of a sequence space \(E\) of complex terms is defined as

\[ E^\alpha = \{(y_k) \in w : (x_k y_k) \in \ell_1 \text{ for all } (x_k) \in E\}. \]

For any normed space \(E\), the set of all continuous linear functions on \(E\) is called its continuous dual, denoted by \(E^*\). Clearly \(E^*\) is a linear space.

Throughout this section we take \(E_k\)'s to be normed linear spaces, normed by \(\|\cdot\|_{E_k}\) for all \(k \in N\). Then the Köthe-Toeplitz dual of \(Z(E_k)\) is defined as

\[ [Z(E_k)]^\alpha = \{(y_k) : y_k \in E_k^* \text{ for all } k \in N \text{ and } (\|x_k\|_{E_k} \|y_k\|_{E_k^*}) \in \ell_1\}. \]

The following well-known result will be used for establishing the result of this section.

\textbf{Lemma 3.} Let \(p_k > 0\) for all \(k \in N\). Then

(i) \([c_0(p)]^\alpha = M_0(p) = \bigcup_{J \in N} \{(a_k) : \sum_{k=1}^{\infty} |a_k| J^{-t_k} < \infty\}\).

(ii) \([\ell_\infty(p)]^\alpha = M_\infty(p) = \bigcap_{J \in N} \{(a_k) : \sum_{k=1}^{\infty} |a_k| J^{t_k} < \infty\}\).

\textbf{Theorem 8.} Let \(p_k > 0\) for all \(k \in N\). Then

(i) \([c_0(E_k, \Lambda, p)]^\alpha = \bigcup_{J \in N} \{(a_k) : \sum_{k=1}^{\infty} \|\gamma_k^{-1} a_k\|_{E_k^*} J^{-t_k} < \infty\}\).
(ii) $[\ell_\infty(E_k, \Lambda, p)]^\alpha = \bigcap_{J \in \mathbb{N}} \{ (a_k) : \sum_{k=1}^{\infty} \| \gamma_k^{-1} a_k \|_{E_k^*} J^k < \infty \}$.

(iii) $[c_0(E_k, \Lambda, p)]^\alpha = \bigcup_{J \in \mathbb{N}} \{ (a_k) : \sum_{k=1}^{\infty} p_k^{-1} \| a_k \|_{E_k^*} J^k < \infty, \text{ for } r > 0 \}$.

(iv) $[\ell_\infty(E_k, \Lambda, p)]^\alpha = \bigcap_{J \in \mathbb{N}} \{ (a_k) : \sum_{k=1}^{\infty} p_k^{-1} \| a_k \|_{E_k^*} J^k < \infty, \text{ for } r > 0 \}$.

**Proof.** The proof follows from Lemma 3 and the following expressions:

$$\sum_{k=1}^{\infty} \| x_k \|_{E_k} \| a_k \|_{E_k^*} = \sum_{k=1}^{\infty} \| \gamma_k x_k \|_{E_k} \| \gamma_k^{-1} a_k \|_{E_k^*}$$

$$= \sum_{k=1}^{\infty} \| r \gamma_k x_k t_k \|_{E_k} \| r^{-1} \gamma_k^{-1} p_k t_k a_k \|_{E_k^*}$$

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**References**


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