ON STATISTICAL CONVERGENCE OF GENERALIZED DIFFERENCE SEQUENCES

BY

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Abstract. In this paper we define the sequence space \( \Delta^m_v(X) = \{ x \in w : (\Delta^m v x_k) \in X \} \), \( (m \in \mathbb{N}) \), where \( \Delta^m v x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k+i} v_{k+i} \) and \( X \) is any sequence space. We give some relations related to this space. Furthermore we introduce the concept of \( \Delta^m_v \)-statistical convergence and give some inclusion relation between \( \Delta^m_v (w_p) \)-convergence and \( \Delta^m_v \)-statistical convergence.

1. Introduction

Let \( \ell_\infty, c \) and \( c_0 \) be the linear spaces of bounded, convergent and null sequences \( x = (x_k) \) with complex terms, respectively, normed by \( \|x\|_\infty = \sup_k |x_k| \), where \( k \in \mathbb{N} = \{1, 2, \ldots\} \), the set of positive integers. Kizmaz [11] introduced the notion of difference sequence spaces as follows:

\[
\Delta (X) = \{ x = (x_k) : (\Delta x_k) \in X \},
\]
for \( X = \ell_\infty, c \) and \( c_0 \), where \( \Delta x = (x_k - x_{k+1}) \).

Later on the notion was generalized by Et and Çolak [4].

Subsequently difference sequence spaces have been studied by Çolak and Et [2], Et [5], Et and Esi [6] and Et and Nuray [7].

The notion of statistical convergence was introduced by Fast [8] and was studied by Connor [1], Fridy [9], Tripathy [12], Salat [13] and many others.

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The idea depends on the density of subsets of the set $\mathbb{N}$ of natural numbers. A subset $E$ of $\mathbb{N}$ is said to have density $\delta(E)$ if
\[
\delta(E) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k)
\]
exists, where $\chi_E$ is the characteristic function of $E$.

A sequence $(x_k)$ is said to be statistically convergent to $L$ if for every $\varepsilon > 0$,
\[
\delta(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}) = 0.
\]
In this case we write $x_k \to L$ or stat-$\lim x_k = L$.

A sequence space $X$ is said to be solid (or normal) if $(\alpha_k x_k) \in X$, whenever $(x_k) \in X$ for all sequences $(\alpha_k)$ of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$.

A sequence space $X$ is said to be symmetric if $(x_k) \in X$ implies $(x_{\pi(k)}) \in X$, where $\pi(k)$ is a permutation of $\mathbb{N}$.

A sequence algebra is a subspace $X$ of $w$ such that $X$ is closed under multiplication ([10]).

The definition of $v-$invariance of a sequence space $X$ was given by Çolak [3] as follows:

A sequence space $X$ is $v-$invariant if $X_v = X$, where $X_v = \{x = (x_k) : (v_k x_k) \in X\}$.

**Definition 1.1.** We say that the sequence space $\Delta^m_v(X)$ is $v-$invariant if $\Delta^m_v(X) = \Delta^m(X)$.

2. A New Sequence Space

In this section we define the sequence space $\Delta^m_v(X)$ as follows:

\[
\Delta^m_v(X) = \{x = (x_k) : (\Delta^m_v x_k) \in X\},
\]
where, $m \in \mathbb{N}$, $\Delta^m_v x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k+i} v_{k+i}$ and $X$ is any sequence space. Now we give some relations between $\Delta^m_v(X)$ and $X$, and discuss some properties of $\Delta^m_v(X)$.

It is trivial that if $X$ is a linear space, then $\Delta^m_v(X)$ is also a linear space.

**Lemma 2.1.**

i) If $X \subset Y$, then $\Delta^m_v(X) \subset \Delta^m_v(Y)$ and the inclusion is strict.
ii) If \( n < m \), then \( \Delta^n_v (X) \subset \Delta^m_v (X) \) and the inclusion is strict.

**Proof.** The part of proof \( \Delta^m_v (X) \subset \Delta^m_v (Y) \) and \( \Delta^n_v (X) \subset \Delta^m_v (X) \) are easy. To show the inclusions are strict, choose \( X = c, Y = \ell_\infty \) and \( x = (1, 0, 1, 0, \ldots), v = (1, 1, \ldots) \) then \( x \in \Delta^n_v (\ell_\infty) \setminus \Delta^m_v (c) \). If we choose \( x = (k^m) \) and \( v = (1, 1, \ldots) \), then \( x \in \Delta^m (c) \setminus \Delta^{m-1} (c) \).

The proof of the following theorem are obtained by using the same technique of Et and Nuray [7, Theorem 2.2], therefore we give it without proof.

**Theorem 2.2.** If \( X \) is a Banach space normed by \( \| \cdot \| \), then \( \Delta^m_v (X) \) is also a Banach space normed by

\[
\| x \|_v = \sum_{i=1}^{m} |x_i v_i| + \| \Delta^m_v (x) \|,
\]

where \( v_k \neq 0 \) for all \( k \in \mathbb{N} \).

Since the space \( \Delta^m_v (X) \) is Banach space with continuous coordinates, that is, \( \| x^s - x \|_v \to 0 \) implies \( |x^s_k - x_k| \to 0 \) for each \( k \in \mathbb{N} \), as \( s \to \infty \), it is BK-space.

Let \( X \subset \ell_\infty \) and \( X \) is a Banach space normed by \( \| \cdot \| \). Let us define the operator

\[
D : \Delta^m_v (X) \to \Delta^m_v (X),
\]

as \( Dx = (0, 0, \ldots, 0, x_{m+1}, x_{m+2}, \ldots) \), where \( x = (x_1, x_2, \ldots) \). It is trivial that \( D \) is a bounded linear operator on \( \Delta^m_v (X) \). The space

\[
D (\Delta^m_v (X)) = D \Delta^m_v (X) = \{ x = (x_k) : x \in \Delta^m_v (X), x_1 = x_2 = \ldots = x_m = 0 \},
\]

is a subspace of \( \Delta^m_v (X) \) and \( \| x \|_v = \| \Delta^m_v (x) \| \).

\( D\Delta^m_v (X) \) and \( X \) are equivalent as topological space since

\[
\Delta^m_v : D\Delta^m_v (X) \to X, \text{ defined by } \Delta^m_v (x) = y = (\Delta^m_v x_k),
\]

is a linear homeomorphism.

**Theorem 2.3.** Let \( X \) be a Banach space and \( A \subset X \). Then

i) If \( A \) is a closed set in \( X \), then \( \Delta^m_v (A) \) is also a closed set in \( \Delta^m_v (X) \).

ii) If \( A \) is a convex set in \( X \), then \( \Delta^m_v (A) \) is also a convex set in \( \Delta^m_v (X) \).
iii) If \( A \) is a separable space, then \( \Delta^m_v (A) \) is also a separable space.

iv) If \( A \) is a nowhere dense set in \( X \), then \( \Delta^m_v (A) \) is a nowhere dense set in \( \Delta^m_v (X) \).

v) If \( A \) is a dense set in \( X \), then \( \Delta^m_v (A) \) is a dense set in \( \Delta^m_v (X) \).

vi) \( \Delta^m_v (w) = w \).

Proof. (i) Since \( A \subset X \), then \( \Delta^m_v (A) \subset \Delta^m_v (X) \) by Lemma 2.1. Now we show that \( \Delta^m_v (A) = \Delta^m_v (\overline{A}) \), where \( \overline{A} \) is the closure of \( A \). Let \( x \in \Delta^m_v (A) \), then there exists a sequence \((x_n)\) in \( \Delta^m_v (A) \) such that

\[
\| (x_k^n) - (x_k) \|_{\Delta^m_v} \to 0, \quad n \to \infty,
\]

in \( \Delta^m_v (A) \), and so

\[
\sum_{i=1}^{m} |x_i^n - x_i| + \| \Delta^m_v (x_k^n) - \Delta^m_v (x_k) \| \to 0, \quad n \to \infty,
\]

in \( A \). Thus \( x = \lim_{n \to \infty} x^n \) exists in \( \Delta^m_v (\overline{A}) \). Conversely if \( x \in \Delta^m_v (\overline{A}) \), then \( x \in \Delta^m_v (A) \). Let \( x \in \Delta^m_v (A) \), then there exists a sequence \((x^n)\) in \( \Delta^m_v (A) \) such that

\[
\| (x_k^n) - (x_k) \|_{\Delta^m_v} \to 0, \quad n \to \infty,
\]

in \( \Delta^m_v (A) \), and so

\[
\sum_{i=1}^{m} |x_i^n - x_i| + \| \Delta^m_v (x_k^n) - \Delta^m_v (x_k) \| \to 0, \quad n \to \infty,
\]

in \( A \). Thus \( x = \lim_{n \to \infty} x^n \) exists in \( \Delta^m_v (\overline{A}) \). Since \( A \) is closed, \( x \in \Delta^m_v (\overline{A}) \). Hence \( x \in \Delta^m_v (\overline{A}) \). This completes the proof.

ii) Proof follows from the following equality

\[
\lambda \Delta^m_v x + (1 - \lambda) \Delta^m_v y = \Delta^m_v (\lambda x + (1 - \lambda) y), \quad (0 \leq \lambda \leq 1).
\]

The proof of (iii) is similar to that of (i).

iv) Suppose that \( \overline{A} = \phi \), but \( \Delta^m_v (\overline{A}) \neq \phi \). Then \( \overline{A} \) contains no neighborhood and \( B(a) \subset \Delta^m_v (\overline{A}) \), where \( B(a) \) is a neighborhood of center \( a \) and radius \( r \). Hence \( a \in B(a) \subset \Delta^m_v (\overline{A}) \). This implies that \( \Delta^m_v (a) \in \overline{A} \). So \( B(\Delta^m_v (a)) \cap A \neq \phi \). This contradicts to \( \overline{A} = \phi \). Hence \( \Delta^m_v (A) = \phi \).

(v) and (vi) are trivial.

Corollary 2.4. Let \( m \) be a positive integer space. Then \( \Delta^m_v (w) = \Delta^m (w) \), that is \( \Delta^m (w) \) is \( v \)-invariant.

It is well known that \( c_0 \) is a sequence algebra and a normal space.

In general it is difficult to predict about the sequence algebra, the solidity and the symmetricity of \( \Delta^m_v (X) \) when \( m > 0 \).
**Theorem 2.5.** The space $\Delta^m_v(X)$ is not sequence algebra, is not solid, is not symmetric.

For the proof of the Theorem 2.5 consider the following examples.

**Example 1.** Let $X = c_0$ and $v = (1, 1, \ldots)$. Consider the sequences $x = (k^m)$, $y = (k^{m-1})$. Clearly, $x, y \in \Delta^m(c_0)$, but $x \cdot y \notin \Delta^m_v(c_0)$ for $m \geq 2$.

**Example 2.** Let $X = \ell_\infty$, $v = (1, 1, \ldots)$ and consider the sequences $x = (k^m)$, $\alpha_k = (-1)^k$. Then $(x_k) \in \Delta^m(\ell_\infty)$, but $(\alpha_k x_k) \notin \Delta^m(\ell_\infty)$.

**Example 3.** Let $X = c$ and $v = (1, 1, \ldots)$. Consider the sequence $x = (k^m)$.

Let $(y_k)$ be a rearrangement of $(x_k)$, which is defined as follows:

$$y_k = (x_1, x_2, x_4, x_3, x_9, x_5, x_{16}, x_6, x_{25}, x_7, x_{36}, x_8, x_{49}, x_{10}, \ldots).$$

Then $(x_k) \in \Delta(c)$, but $(y_k) \notin \Delta(c)$.

In the case $X = w_p$, we obtain the following sequence

$$\Delta^m_v(w_p) = \left\{ x = (x_k) : \frac{1}{n} \sum_{k=1}^{n} |\Delta^m_v x_k| \to 0, \ n \to \infty, \ p > 0, \ for \ some \ L \right\}.$$

If $x \in \Delta^m_v(w_p)$, then we say that $x$ is strongly $\Delta^m_v$–Cesàro summable to $L$ and we shall write $x_k \to L (\Delta^m_v(w_p))$.

**Theorem 2.6.** The sequence space $\Delta^m_v(w_p)$ is a Banach space for $1 \leq p < \infty$ normed by

$$\|x\|_{\Delta^1_v} = \sum_{i=1}^{m} |v_i x_i| + \sup_n \left( \frac{1}{n} \sum_{k=1}^{n} |\Delta^m_v x_k|^p \right)^{\frac{1}{p}},$$

and a complete $p$–normed space for $0 < p < 1$, $p$–normed by

$$\|x\|_{\Delta^2_v} = \sum_{i=1}^{m} |v_i x_i|^p + \sup_n \frac{1}{n} \sum_{k=1}^{n} |\Delta^m_v x_k|^p.$$
3. Statistical Convergence

In this section we give $\Delta^m_v$—statistical convergence and inclusion theorems between the space $\Delta^m_v(S)$ and other some sequence spaces. In this section, we dropped the condition $v_k \neq 0$, for all $k \in \mathbb{N}$.

**Definition 3.1.** A sequence $(x_k)$ is said to be $\Delta^m_v$—statistically convergent if there is a complex number $L$ such that
\[
\lim_{n \to \infty} \frac{1}{n} |\{k \leq n : |\Delta^m_v x_k - L| \geq \varepsilon\}| = 0,
\]
for every $\varepsilon > 0$. 

In this case we write $x_k \to L(\Delta^m_v(S))$. In the case $m = 0$, we shall write $S_v$.

**Definition 3.2.** A sequence $(x_k)$ is said to be $\Delta^m_v$—statistically Cauchy if for any given $\varepsilon > 0$, there exists a number $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that
\[
\lim_{n \to \infty} \frac{1}{n} |\{k \leq n : |\Delta^m_v x_k - \Delta^m_v x_{n_0}| \geq \varepsilon\}| = 0.
\]

It can be shown that if $x$ is a $\Delta^m_v$—statistically convergent sequence, then $x$ is a $\Delta^m_v$—statistically Cauchy sequence.

The proof of the following two theorems are obtained by using the same technique of Theorem 4.2 and Theorem 4.7 of Et and Nuray [7], therefore we give them without proofs.

**Theorem 3.3.** Let $0 < p < \infty$. Then
i) $x_k \to L(\Delta^m_v(w_\infty))$, then $x_k \to L(\Delta^m_v(S))$.
ii) $x_k \to L(\Delta^m_v(\ell_\infty))$ and $x_k \to L(\Delta^m_v(S))$, then $x_k \to L(\Delta^m_v(w_\infty))$.
iii) $\Delta^m_v(\ell_\infty) \cap \Delta^m_v(S) = \Delta^m_v(\ell_\infty) \cap \Delta^m_v(w_\infty)$.

**Theorem 3.4.** If $x$ is a sequence for which there is a $\Delta^m_v$—statistically convergent sequence $y$ such that $\Delta^m_v x_k = \Delta^m_v y_k$ for almost all $k$. Then $x$ is $\Delta^m_v$—statistically convergent sequence.

**Theorem 3.5.**

i) $\Delta^m_v(c) \subset \Delta^m_v(S)$ and the inclusion is strict.
ii) $S_v \subset \Delta^m_v(S)$ and the inclusion is strict.
iii) $\Delta^m_v(S)$ and $\Delta^m_v(\ell_{\infty})$ overlap but neither one contains the other.

iv) $\Delta^m_v(S)$ and $\ell_{\infty}$ overlap but neither one contains the other.

v) $S$ and $\Delta^m_v(S)$ overlap but neither one contains the other.

vi) $S$ and $\Delta^m_v(c)$ overlap but neither one contains the other.

vii) $S$ and $\Delta^m_v(c_0)$ overlap but neither one contains the other.

viii) $S$ and $\Delta^m_v(\ell_{\infty})$ overlap but neither one contains the other.

ix) $S_v$ and $\Delta^m_v(S)$ overlap but neither one contains the other.

x) $S_v$ and $\Delta^m_v(\ell_{\infty})$ overlap but neither one contains the other.

xi) $S_v$ and $\Delta^m_v(c)$ overlap but neither one contains the other.

xii) $S_v$ and $\Delta^m_v(c_0)$ overlap but neither one contains the other.

xiii) $S_v$ and $\Delta^m_v(c_0)$ overlap but neither one contains the other.

xiv) $S_v$ and $\ell_{\infty}$ overlap but neither one contains the other.

xv) $\Delta^m_v(S)$ and $\Delta^m_v(S)$ overlap but neither one contains the other.

xvi) $\Delta^m_v(S)$ and $\Delta^m_v(\ell_{\infty})$ overlap but neither one contains the other.

Proof. i) Since $c \subset S$, then $\Delta^m_v(c) \subset \Delta^m_v(S)$. Choose the sequence $x = (x_k)$ such that

$$\Delta^m_v x = \begin{cases} \sqrt{k}, & k = n^2, n = 1, 2, \ldots, \\ 0, & k \neq n^2. \end{cases}$$

Then we obtain $\Delta^m_v x \in S$ but $\Delta^m_v x \notin c$.

ii) Let $x_k \to L(S_v)$. Then the inclusion follows from the inequality

$$|\Delta_v x_k| \leq |v_k x_k - L| + |v_{k+1} x_{k+1} - L|$$

for almost all $k \in \mathbb{N}$.

To show that the inclusion is strict, let us take $x = (k)$ and $v = (1, 1, \ldots)$, then $x \notin S_v$, but $x \in \Delta^m_v(S)$.

Since the sequence $x = (0)$ belongs to all the above sequence spaces, the overlapping part of the proof for the cases (iii) to (xvi) is obvious. For the other parts in each case, consider the following examples.

iii) Consider the sequence defined by (3.1). Then $x \in \Delta^m_v(S)$, but $x \notin \Delta^m_v(\ell_{\infty})$. Conversely if we choose $x = (1, 0, 1, 0, \ldots)$ and $v = (1, 1, \ldots)$, then $x \in \Delta^m_v(\ell_{\infty})$, but $x \notin \Delta^m_v(S)$.

iv) The sequences $x = (x_k)$ and $v = (v_k)$ defined by

$$x_k = \begin{cases} \sqrt{k}, & k = n^2, \\ 0, & k \neq n^2, \end{cases} \quad n = 1, 2, \ldots,$$
and
\[
    v_k = \begin{cases} 
        0, & k = n^2, \\
        1, & k \neq n^2, 
    \end{cases}, \quad n = 1, 2, \ldots \tag{3.3}
\]

Then \( x \notin \ell_\infty \) and \( x \in \Delta^m_v(S) \). For the converse, consider the sequences \( x = (1, 1, 1, \ldots) \) and \( v = (k^{m+1}) \). Then \( x \in \ell_\infty \), but \( x \notin \Delta^m_v(\ell_\infty) \), where \( \Delta^m_v(x) = (-1)^{m+2}(m+1)!\left(k + \frac{m}{2}\right) \).

\( \text{v)} \) The sequence \( x = (1, 1, \ldots) \) is statistically convergent. If we take \( v = (1, 0, 1, \ldots) \), then \( x \notin \Delta^m_v(S) \), where \( \Delta^m_v(x) = (-1)^{k+1}2^{m-1} \).

Conversely the sequence \( x = (k) \) is not statistically convergent. If we take \( v = (1, 1, 1, \ldots) \), then \( x \in \Delta^m_v(S) \).

\( \text{vi)} \) Let \( x = (1, 1, \ldots) \) and \( v = (1, 0, 1, \ldots) \). Then \( x \in S \) but \( x \notin \Delta^m_v(c) \). For the converse, let \( x = (k) \) and \( v = (1, 1, \ldots) \), then \( x \notin S \), but \( x \in \Delta^m_v(c) \).

\( \text{vii)} \) The proof is the same as vi).

\( \text{viii)} \) Consider the sequence defined by (3.2) and \( v = (1, 1, \ldots) \). Then \( x \in S \), but \( x \notin \Delta^m_v(\ell_\infty) \). Now we consider the sequence \( x = (k) \) and \( v = (1, 1, 1, \ldots) \), then \( x \notin S \), but \( x \in \Delta^m_v(\ell_\infty) \).

\( \text{ix)} \) Let \( x = (k) \) and \( v = (1, 1, 1, \ldots) \), then \( x \notin S_v \), but \( x \in \Delta^m(S) \).

\( \iff \) Let \( x = (1, 0, 1, 0, \ldots) \) and \( v = (0, 1, 0, 1, \ldots) \), then \( x \in S_v \), but \( x \notin \Delta^m(S) \).

\( \text{x)} \) Consider the sequence \( x = (x_k) \) defined by (3.2) and let \( v = (1, 1, 1, \ldots) \). Then \( x \in S_v \), but \( x \notin \Delta^m_v(\ell_\infty) \).

\( \iff \) Let \( x = (k) \) and \( v = (k) \), then \( x \notin S_v \), but \( x \in \Delta^m(\ell_\infty) \).

The proof of (x), (xii) and (xiii) are the same as (x).

\( \text{xiv)} \) Consider the sequences defined by (3.2) and (3.3). Then \( x \in S_v \), but \( x \notin \ell_\infty \).

\( \iff \) Let \( x = (1, 0, 1, 0, \ldots) \) and \( v = (1, 1, 1, \ldots) \), then \( x \notin \ell_\infty \), but \( x \notin S_v \).

\( \text{xv)} \) Let \( x = (k^{m+1}) \) and \( v = (k^{-2}) \). Then \( x \notin \Delta^m(S) \), but \( x \in \Delta^m_v(S) \).

Conversely let \( x = (k^m) \) and \( v = (k) \), then \( x \in \Delta^m(S) \), but \( x \notin \Delta^m_v(S) \).

\( \text{xvi)} \) Let \( x = (k^{m+1}) \) and \( v = (k^{-2}) \). Then \( x \in \Delta^m_v(S) \), but \( x \notin \Delta^m_v(\ell_\infty) \).

\( \iff \) Let \( x = (1, 0, 1, 0, \ldots) \) and \( v = (1, 1, \ldots) \), then \( x \in \Delta^m(\ell_\infty) \), but \( x \notin \Delta^m_v(S) \).
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References


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