WHAT CAN WE DO WITH NASH’S EMBEDDING THEOREM?

BY

BANG-YEN CHEN

Abstract. According to the celebrated embedding theorem of J. F. Nash, every Riemannian manifold can be isometrically embedded in some Euclidean spaces with sufficiently high codimension. An immediate problem concerning Nash’s theorem is the following:

Problem: What can we do with Nash’s embedding theorem? In other words, what can we do with arbitrary Euclidean submanifolds of arbitrary high codimension if no local or global assumption were imposed on the submanifold?

In this survey, we present some general optimal solutions to this and related problems. We will also present many applications of the solutions to the theory of submanifolds as well as to Riemannian geometry.

1. What Can We Do with Nash’s Embedding Theorem?

According to the celebrated embedding theorem of J. F. Nash [65] published in 1956, every Riemannian manifold can be isometrically embedded in some Euclidean spaces with sufficiently high codimension.
An immediate question concerning Nash’s embedding theorem is the following.

Problem 1. What can we do with Nash’s embedding theorem? In other words, what can we do with arbitrary Euclidean submanifolds of arbitrary high codimension if “no local or global assumption” were imposed on the submanifold?

The Nash theorem was aimed for in the hope that if Riemannian manifolds could always be regarded as Riemannian submanifolds, this would then yield the opportunity to use extrinsic help. Till when observed as such by M. Gromov [52] this hope had not been materialized however.

The main reason for this is due to the lack of controls of the extrinsic properties of the submanifolds by the known intrinsic data (see [80]).

In order to overcome such difficulty, one needs to introduce new type of Riemannian invariants and to establish general sharp relationships between extrinsic properties of the submanifolds with the new type of intrinsic Riemannian invariants on the submanifolds.

In other words, one needs to establish some general optimal solutions to the following fundamental problem:

Problem 2.

∀ $M$ \text{isometric immersion} $\mathbb{E}^m \Rightarrow \begin{cases} \text{Sharp relationship between intrinsic} \\ \text{and extrinsic invariants}\end{cases}$

Notice that we do not impose any intrinsic or extrinsic assumption on the Riemannian submanifolds in Problem 2.

In this survey, we will present optimal general solutions to these two fundamental problems and also solutions to some closely related problems. Many immediate applications of these optimal general solutions will also be presented in this survey. Most of these will be presented in the first seven sections.

In sections 8 and 9 we explain how to assign a number, called the contact number to an arbitrary Euclidean submanifold. The contact number of a Euclidean submanifold is either a natural number or $+\infty$, which measures the degree of contact of geodesics and normal sections of the submanifold. In these
two sections, we will explain the relationships between contact number with the
notions of isotropic submanifolds, holomorphic curves and special Lagrangian
surfaces.

In section 10, we present an application of the study of contact number to
pseudo-umbilical immersions; namely, to obtain the first non-trivial examples of
pseudo-umbilical submanifolds in Euclidean spaces.

In the last section, we present some sharp results for Kählerian submanifolds
which can be regarded as the Kählerian version of a solution to Problem 2 for
the Euclidean submanifolds given in earlier section.

2. Some Solutions to Problems 1 and 2

In order to provide some general optimal solutions to Problems 1 and 2, one
needs to introduce a new type of scalar-valued curvature invariants, denoted by
\( \delta(n_1, \ldots, n_k) \), which are obtained from the scalar curvature (which is the total
sum of sectional curvatures) by deleting a portion of sectional curvatures. (See
[15, 17] and also [11, 12] for details). This was done as follows:

Let \( M \) be a Riemannian \( n \)-manifold. Denote by \( K(\pi) \) the sectional curvature
of \( M \) associated with a plane section \( \pi \subset T_p M, p \in M \). For any orthonormal
basis \( e_1, \ldots, e_n \) of the tangent space \( T_p M \), the scalar curvature \( \tau \) at \( p \) is defined
to be

\[
\tau(p) = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j).
\]

(2.1)

Let \( L \) be a subspace of \( T_p M \) of dimension \( r \geq 2 \) and \( \{e_1, \ldots, e_r\} \) an orthonor-
mal basis of \( L \). The scalar curvature \( \tau(L) \) of the \( r \)-plane section \( L \) is defined by

\[
\tau(L) = \sum_{1 \leq \alpha < \beta \leq r} K(e_\alpha \wedge e_\beta), \quad 1 \leq \alpha, \beta \leq r.
\]

(2.2)

The scalar curvature \( \tau(p) \) of \( M \) at a point \( p \in M \) is nothing but the scalar
curvature of the tangent space of \( M \) at \( p \). And if \( L \) is a 2-plane section, \( \tau(L) \) is
nothing but the sectional curvature \( K(L) \) of \( L \). In general, \( \tau(L) \) with \( \dim L \geq 2 \)
is nothing but the scalar curvature of the image \( \exp_p(L) \) of \( L \) at \( p \) under the
exponential map at \( p \).
For any integer \( n \geq 2 \), we denote by \( S(n) \) the finite set consisting of all \( k \)-tuples \((n_1, \ldots, n_k)\) of integers \( n_1, \ldots, n_k \geq 2 \) with \( n_1 + \cdots + n_k \leq n \), where \( k \) is a non-negative integer.

The cardinal number \(#S(n)\) of \( S(n) \) is equal to \( p(n) - 1 \), where \( p(n) \) denotes the number of partition of \( n \) which increases quite rapidly with \( n \). For instance, for

\[
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots, 50, \ldots, 100, \ldots, 200,
\]

the cardinal number \(#S(n)\) are given respectively by

\[
1, 2, 4, 6, 10, 14, 21, 29, 41, \ldots, 204225, \ldots, 190569291, \ldots, 3972999029387.
\]

The asymptotic behavior of \(#S(n)\) is given by \(#S(n) \sim (1/(4n\sqrt{3}) \exp[\pi\sqrt{2n/3}]\) as \( n \to \infty \).

For each \( k \)-tuple \((n_1, \ldots, n_k)\) in \( S(n) \), the author introduced a decade ago the Riemannian invariant \( \delta(n_1, \ldots, n_k) \) by

\[
\delta(n_1, \ldots, n_k) = \tau - \inf \{ \tau(L_1) + \cdots + \tau(L_k) \}
\]

(2.3)

where \( L_1, \ldots, L_k \) run over all \( k \) mutually orthogonal subspaces of \( T_pM \) such that \( \dim L_j = n_j, j = 1, \ldots, k \).

The following general optimal inequalities from \([15, 17]\) provide sharp solutions to both Problem 1 and Problem 2.

**Theorem 1.** For any Euclidean submanifold \( M \) and for any \((n_1, \ldots, n_k) \in S(n), n = \dim M\), we have the following inequalities:

\[
\delta(n_1, \ldots, n_k) \leq c(n_1, \ldots, n_k)H^2,
\]

(2.4)

where \( H^2 \) is the squared mean curvature and \( c(n_1, \ldots, n_k) \) are positive numbers given by

\[
c(n_1, \ldots, n_k) = \frac{n^2(n + k - 1 - \sum n_j)}{2(n + k - \sum n_j)}.
\]

(2.5)

The equality case of inequality (2.4) holds at a point \( p \in M \) if and only if, there exists an orthonormal basis \( e_1, \ldots, e_n \) at \( p \), such that the shape operators of \( M \) with respect to \( e_1, \ldots, e_n \) at \( p \) take the following form:

\[
A_\xi = \begin{pmatrix}
A_1^\xi & 0 \\
\cdot & \cdot \\
0 & A_k^\xi
\end{pmatrix},
\]

(2.6)
where \( \{ A_j^\xi \}_{j=1}^k \) are symmetric \( n_j \times n_j \) submatrices satisfying

\[
\text{trace } (A_j^\xi) = \cdots = \text{trace } (A_k^\xi) = \mu_\xi.
\] (2.7)

Inequalities (2.4) give prima controls on the most important extrinsic curvature, the squared mean curvature \( H^2 \), by the initial intrinsic curvatures, the \( \delta \)-invariants \( \delta(n_1, \ldots, n_k) \), of the Riemannian manifold.

In general, there do not exist direct relationship between the newly defined \( \delta \)-invariants \( \delta(n_1, \ldots, n_k) \). On the other hand, we have the following.

**Maximum Principle.** Let \( M \) be an \( n \)-dimensional submanifold of a Euclidean \( m \)-space \( \mathbb{E}^m \). If it satisfies the equality case of (2.4), i.e., it satisfies

\[
H^2 = \Delta(n_1, \ldots, n_k)
\] (2.8)

for a \( k \)-tuple \( (n_1, \ldots, n_k) \in S(n) \), then

\[
\Delta(n_1, \ldots, n_k) \geq \Delta(m_1, \ldots, m_j)
\] (2.9)

for any \( (m_1, \ldots, m_j) \in S(n) \), where

\[
\Delta(n_1, \ldots, n_k) = \frac{\delta(n_1, \ldots, n_k)}{c(n_1, \ldots, n_k)}.
\] (2.10)

For any isometric immersion \( x : M \to \mathbb{E}^m \) of a Riemannian \( n \)-manifold \( M \) in \( \mathbb{E}^m \). Theorem 1 yields

\[
H^2(p) \geq \hat{\Delta}_0(p),
\] (2.11)

where \( \hat{\Delta}_0 \) is the Riemannian invariant on \( M \) defined by

\[
\hat{\Delta}_0 = \max \{ \Delta(n_1, \ldots, n_k) : (n_1, \ldots, n_k) \in S(n) \}.
\]

Inequality (5.4) enables us to introduce the notion of ideal immersions as given in the following.

**Definition 1.** An isometric immersion of a Riemannian \( n \)-manifold \( M \) in \( \mathbb{E}^m \) is called an ideal immersion if it satisfies the equality case of (2.11) identically.

The above maximum principle yields the following important fact:
If an isometric immersion $x : M \to \mathbb{E}^m$ satisfies equality (2.4) for a given $k$-tuple $(n_1, \ldots, n_k) \in S(n)$, then it is an ideal immersion automatically.

By applying the inequalities (2.4) and the theory of finite type submanifolds ([10, 14]), we can establish the following new intrinsic results concerning intrinsic spectral properties of homogeneous spaces via extrinsic data.

**Theorem 2.** If $M$ is a compact homogeneous Riemannian $n$-manifold with irreducible isotropy action, then the first nonzero eigenvalue $\lambda_1$ of the Laplacian on $M$ satisfies

$$\lambda_1 \geq n \Delta(n_1, \ldots, n_k)$$

for any $k$-tuple $(n_1, \ldots, n_k) \in S(n)$.

The equality sign of (2.12) holds if and only if $M$ admits a 1-type ideal immersion in a Euclidean space. Here 1-type immersion is in the sense of [10, 14].

**Remark 1.** In particular, if $k = 0$, inequality (2.12) reduces to the well-known result of T. Nagano on $\lambda_1$ obtained in [64]; namely

$$\lambda_1 \geq n \rho,$$

where $\rho = \tau/(\binom{n}{2})$ is the normalized scalar curvature.

In general, we have $\Delta(n_1, \ldots, n_k) \geq \rho$. Moreover, we have $\Delta(n_1, \ldots, n_k) > \rho$ for $k > 0$ on most Riemannian manifolds.

**Remark 2.** It seems to the author that it is impossible to obtain the intrinsic result, inequality (2.12), without the help from the theory of submanifolds via Nash’s embedding theorem.

This example shows that, contrast to Gromov’s remark on Nash’s theorem (see the first section or [52]), indeed the original aim of Nash’s embedding theorem can be fulfilled, in other words, it is possible to establish intrinsic results on Riemannian manifolds with extrinsic help via Nash’s embedding theorem.

There do exist many other applications of Theorem 1. Here we give a few of them (see [15, 17] for details).
Corollary 1. Let $M$ be a Riemannian $n$-manifold. If there is a $k$-tuple $(n_1, \ldots, n_k)$ in $S(n)$ and a point $p \in M$ such that $\delta(n_1, \ldots, n_k) > 0$ at $p$, then $M$ admits no minimal isometric immersion in any Euclidean space regardless of codimension.

Remark 3. It is well-known that the equation of Gauss implies immediately that, for each minimal submanifold in a Euclidean space, the Ricci tensor satisfies $\text{Ric} \leq 0$. For many years this was the only known general necessary Riemannian condition for a Riemannian manifold to admit a minimal isometric immersion in Euclidean spaces. However, Corollary 1 provides us infinitely many new obstructions to isometric minimal immersions of Riemannian manifolds into Euclidean spaces.

Corollary 2. Let $M_1^{n_1}, \ldots, M_k^{n_k}$ be Riemannian manifolds of dimensions $\geq 2$ with scalar curvatures $\leq 0$. Then every minimal isometric immersion

$$f : M_1^{n_1} \times \cdots \times M_k^{n_k} \times \mathbb{E}^n - \sum n_j \rightarrow \mathbb{E}^m$$

of $M_1^{n_1} \times \cdots \times M_k^{n_k} \times \mathbb{E}^n - \sum n_j$ in any Euclidean $m$-space is a product immersion $f_1 \times \cdots \times f_k \times \iota$ of $k$ minimal immersions $f_j : M_j^{n_j} \rightarrow \mathbb{E}^{m_j}, j = 1, \ldots, k$, and a totally geodesic immersion $\iota$.

For Lagrangian immersions in complex Euclidean $n$-space $\mathbb{C}^n$, a well-known result of Gromov [51] states that a compact $n$-manifold $M$ admits a Lagrangian immersion (not necessary isometric) in $\mathbb{C}^n$ if and only if the complexification $TM \otimes \mathbb{C}$ of the tangent bundle of $M$ is trivial. Gromov’s result implies that there is no topological obstruction to Lagrangian immersions for compact 3-manifolds in $\mathbb{C}^3$, because the tangent bundle of a 3-manifold is trivial.

The following result yields Riemannian obstructions to Lagrangian isometric immersions into complex Euclidean spaces.

Corollary 3. Let $M$ be a compact Riemannian $n$-manifold with finite fundamental group $\pi_1$ or with null first betti number, i.e., $b_1 = 0$. If there is a $k$-tuple $(n_1, \ldots, n_k) \in S(n)$ such that $\delta(n_1, \ldots, n_k) > 0$, then $M$ admits no Lagrangian isometric immersion in any complex $n$-torus or in complex Euclidean $n$-space.
Remark 4. Similar results hold for arbitrary submanifolds in Riemannian manifold of constant sectional curvature with arbitrary codimension.

Remark 5. Since the $\delta$-invariants $\delta(n_1, \ldots, n_k)$ were introduced a decade ago, inequalities (2.4) have been studied by many geometers, (see for instance, [3, 4, 5, 6, 7, 29, 30, 35, 37, 39, 43, 44, 47, 48, 49, 53, 54, 56, 57, 61, 62, 67, 69, 71, 72, 73, 75, 76, 78, 81], among many others). Moreover, the $\delta$-invariants has been applied to affine geometry in [28, 58, 59, 74].

3. Solutions in Terms of Shape Operator and Ricci Curvature

Besides the squared mean curvature $H^2$, the shape operator $A$ is another important extrinsic invariant for submanifolds. In this section, we present a solution to Problems 1 and 2 in terms of the shape operator.

Let $M$ be a Riemannian $n$-manifold and $L^k$ is a $k$-plane section of $T_x M$, $x \in M$. For each unit vector $X$ in $L^k$, we choose an orthonormal basis $\{e_1, \ldots, e_k\}$ of $L^k$ such that $e_1 = X$. Define $Ric_{L^k}$ of $L^k$ at $X$ by

$$Ric_{L^k}(X) = K_{12} + \cdots + K_{1k},$$

where $K_{ij}$ denotes the sectional curvature of the 2-plane section spanned by $e_i, e_j$. We call $Ric_{L^k}(X)$ a $k$-Ricci curvature of $M$ at $X$ relative to $L^k$. Clearly, the $n$-th Ricci curvature is nothing but the Ricci curvature in the usual sense and the second Ricci curvature coincides with the sectional curvature.

For an integer $k \in [2, n]$, let $\theta_k$ be the Riemannian invariant defined by [16]:

$$\theta_k(x) = \left(\frac{1}{k-1}\right) \inf_{L^k, X} Ric_{L^k}(X), \quad X \in T_x M$$

(3.2)

where $L^k$ runs over $k$-plane sections in $T_x M$ and $X$ runs over unit vectors in $L^k$.

The following result from [13, 16] provides an optimal relationship between the $k$-Ricci curvature and the shape operator for an arbitrary Euclidean submanifold, regardless of codimension.

**Theorem 3.** Let $f : M \to \mathbb{R}^m$ be an isometric immersion of a Riemannian $n$-manifold $M$ into a Euclidean $m$-space with arbitrary codimension. Then, for any integer $k$, $2 \leq k \leq n$, and any point $x \in M$, we have:
(1) If \( \theta_k(x) \neq 0 \), then the shape operator in the direction of the mean curvature vector \( \vec{H} \) satisfies
\[
A \vec{H} > \left( \frac{n-1}{n} \right) \theta_k(x) I \quad \text{at } x,
\]
where \( I \) is the identity map and (3.3) means that \( A \vec{H} - ((n-1)/n)\theta_k(x)I \) is positive-definite.

(2) If \( \theta_k(x) = 0 \), then \( A \vec{H} \geq 0 \) at \( x \).

(3) A unit vector \( X \in T_xM \) satisfies \( A \vec{H} X = ((n-1)/n)\theta_k(x)X \) if and only if \( \theta_k(x) = 0 \) and \( X \) lies in the relative null space at \( x \).

(4) \( A \vec{H} \equiv ((n-1)/n)\theta_k I \) at \( x \) if and only if \( x \) is a totally geodesic point.

The estimate of the eigenvalues of the shape operator \( A \vec{H} \) given by (3.3) is sharp.

Theorem 3 implies the following.

**Corollary 4.** If there is an integer \( k \), \( 2 \leq k \leq n \), such that \( \theta_k(x) > 0 \) (respectively, \( \theta_k(x) \geq 0 \)) for a Riemannian \( n \)-manifold \( M \) at a point \( x \in M \), then, for any isometric immersion of \( M \) into any Euclidean space, each eigenvalue of the shape operator \( A \vec{H} \) is greater than \( (n-1)/n \) (respectively, \( \geq 0 \)), regardless of codimension.

When the Euclidean submanifold is a compact, Theorem 3 implies the following.

**Corollary 5.** If \( M \) is a compact hypersurface of \( \mathbb{E}^{n+1} \) satisfies \( \theta_k \geq 0 \) (respectively, \( \theta_k > 0 \)) for some integer \( k \), \( 2 \leq k \leq n \), then \( M \) is embedded as a convex (respectively, strictly convex) hypersurface in \( \mathbb{E}^{n+1} \).

**Remark 6.** The last corollary implies that if \( M \) is a compact hypersurface of \( \mathbb{E}^{n+1} \) with nonnegative Ricci curvature (respectively, with positive Ricci curvature), then \( M \) is embedded as a convex (respectively, strictly convex) hypersurface in \( \mathbb{E}^{n+1} \). In particular, when \( M \) has constant scalar curvature, \( M \) must be embedded as a hypersphere of \( \mathbb{E}^{n+1} \).
We also have the next solution obtained in [16] to Problems 1 and 2 in terms of the Ricci curvature.

**Theorem 4.** Let \( M \) be an \( n \)-dimensional Euclidean submanifold with arbitrary codimension. Then we have:

1. For each unit tangent vector \( X \in T_pM \), we have
   \[
   \text{Ric}(X) \leq \frac{n^2}{4}H^2,
   \]
   where \( \text{Ric}(X) \) the Ricci curvature of \( M \) at \( X \).
2. If \( H(p) = 0 \), then a unit tangent vector \( X \) at \( p \) satisfies the equality case of (3.4) if and only if \( X \) lies in the relative null space \( N_p \) at \( p \).
3. The equality case of (3.4) holds for all unit tangent vectors at \( p \) if and only if \( p \) is a totally geodesic point or \( n = 2 \) and \( p \) is a totally umbilical point.

4. Solutions for the Family of Warped Product Manifolds

In the same line of thought as Problem 2, the author proposed the following problem in December 2001 at a conference on “Geometry of Submanifolds” held at Tokyo Metropolitan University in honor of Professor K. Ogiue on the occasion of his sixtieth anniversary.

**Problem 3.** Let \( \mathcal{F} \) be the family of Riemannian manifolds endowed with a structure \( \mathcal{S} \). What are the special relationships between the intrinsic invariants on \( M \) in \( \mathcal{F} \) and the main extrinsic invariants of \( M \) if \( M \) were immersed isometrically in an arbitrary Euclidean space (or, more generally, space forms) and if no additional local or global assumption were imposed on the submanifolds?

In order to obtain solutions to this problem, one needs to have some intrinsic invariants which are special for the special family of Riemannian manifolds we choose to investigate.

For example, for the family of warped product manifolds endowed with the warped product structure, the intrinsic invariants which are special on warped product manifolds are the invariants obtained from their warping function.
From this point of view, the following optimal result from [21] provides a solution to Problem 3 for the family of warped product manifolds endowed with their warped product structure.

**Theorem 5** Let $\phi : N_1 \times f N_2 \to R^m(c)$ be an arbitrary isometric immersion of a warped product into a Riemannian $m$-manifold $R^m(c)$ of constant sectional curvature $c$. Then the warping function $f$ satisfies

$$\frac{\Delta f}{f} \leq \frac{(n_1 + n_2)^2}{4n_2}H^2 + n_1c, \quad n_i = \dim N_i, \ i = 1, 2, \quad (4.1)$$

where $H^2$ is the squared mean curvature of $M$ and $\Delta$ is the Laplacian operator of $N_1$.

As immediate applications of Theorem 5, we have the following.

**Corollary 6.** Let $N_1 \times f N_2$ be a warped product whose warping function $f$ is a harmonic function. Then

1. $N_1 \times f N_2$ admits no isometric minimal immersion into a hyperbolic space for any codimension.
2. Every isometric minimal immersion from $N_1 \times f N_2$ into a Euclidean space is a warped product immersion.

**Corollary 7.** If $f$ is an eigenfunction of Laplacian on $N_1$ with eigenvalue $\lambda > 0$, then $N_1 \times f N_2$ does not admits an isometric minimal immersion into a Euclidean space or a hyperbolic space for any codimension.

**Corollary 8.** If $N_1$ is a compact Riemannian manifold, then every warped product $N_1 \times f N_2$ does not admit an isometric minimal immersion into a Euclidean space or a hyperbolic space for any codimension.

**Remark 7.** There exist many minimal submanifolds in Euclidean space which are warped products with harmonic warping function. For example, if $N_2$ is a minimal submanifold of the unit $(m-1)$-sphere $S^{m-1} \subset \mathbb{R}^m$, the minimal cone $C(N_2)$ over $N_2$ with vertex at the origin of $\mathbb{R}^m$ is the warped product $\mathbb{R}_+ \times s N_2$ whose warping function $f = s$ is a harmonic function. Here $s$ is the coordinate function of the positive real line $\mathbb{R}_+$. 
Remark 8. In views of Theorem 5, it is interesting to point out that there do exist isometric minimal immersions from warped products \( \mathbb{N}_1 \times_f \mathbb{N}_2 \) into a hyperbolic space such that the warping function \( f \) is an eigenfunction with negative eigenvalue. For example, \( \mathbb{R} \times e^x \mathbb{E}^{n-1} \) admits an isometric minimal immersion into the hyperbolic space \( H^{n+1} \) of constant sectional curvature \(-1\).

Remark 9. In views of Theorem 5, it is interesting to point out that there do exist many isometric minimal immersions from \( \mathbb{N}_1 \times_f \mathbb{N}_2 \) into Euclidean space with compact \( \mathbb{N}_2 \). For examples, a hypercaternoid in \( \mathbb{E}^{n+1} \) is a minimal hypersurfaces which is isometric to a warped product \( R \times f S^{n-1} \). Also, for any compact minimal submanifold \( \mathbb{N}_2 \) of \( S^{m-1} \subset \mathbb{E}^m \), the minimal cone \( C(\mathbb{N}_2) \) is a warped product \( R_+ \times s \mathbb{N}_2 \) which is also a such example.

Remark 10. Contrast to Euclidean and hyperbolic spaces, the standard \( m \)-sphere \( S^m \) admits warped product minimal submanifolds \( \mathbb{N}_1 \times_f \mathbb{N}_2 \) such that \( \mathbb{N}_1, \mathbb{N}_2 \) are both compact. The simplest such examples are minimal Clifford tori \( M_{k,n-k} \), \( k = 2, \ldots, n-1 \), in \( S^{n+1} \) defined by \( M_{k,n-k} = S^k(\sqrt{k/n}) \times S^{n-k}(\sqrt{n - k/n}) \).

Remark 11. Similar results as inequality (4.1) also hold for warped products in complex hyperbolic \( m \)-space \( CH^m(4c) \) as well as for warped product manifolds in complex projective \( m \)-space \( CP^m(4c) \) (see [22, 23] for the details).

5. A Solution for the Family of Einstein Manifolds

Next, we consider Problem 3 for the important family of Einstein manifolds. In order to obtain a general optimal solution for this family, we need to introduce new Riemannian invariants, denoted by \( \delta^Ric \), defined as follows:

Let \( p \) be a point in a Riemannian \( n \)-manifold \( M \) and \( q \) a natural number \( \leq n/2 \). For a given point \( p \in M \), let \( \pi_1, \ldots, \pi_q \) be \( q \) mutually orthogonal plane sections in \( T_pM \). Define the invariant \( K^{inf}_q(p) \) to be the infimum of the average of the sectional curvatures \( K(\pi_1), \ldots, K(\pi_q) \), i.e.,

\[
K^{inf}_q(p) = \inf_{\pi_1, \ldots, \pi_q} \frac{K(\pi_1) + \cdots + K(\pi_q)}{q},
\]

(5.1)

where \( \pi_1, \ldots, \pi_q \) run over all mutually orthogonal \( q \) plane sections in \( T_pM \).
For a natural number \( q \leq n/2 \), we define the Riemannian invariant \( \delta^R_{q}(p) \) by

\[
\delta^R_{q}(p) = \sup_{X \in T^1_p M} \text{Ric}(X, X) - \frac{2q}{n} K^\text{inf}_q(p),
\]

where \( n = \text{dim} M \) and \( X \) runs over vectors in \( T^1_p M := \{ X \in T_p M : |X| = 1 \} \).

Recall that a submanifold \( M \) of a Riemannian manifold is called \textit{pseudo-umbilical} if its mean curvature vector \( \vec{H} \) is nonzero and its shape operator \( A \vec{H} \) at the mean curvature vector is proportional to the identity map (cf. [8]).

For the family of Einstein manifolds, the following results from [25] provide us some general optimal solutions to Problem 3.

**Theorem 6.** For any integer \( k \geq 2 \) and any isometric immersion of an Einstein \( 2k \)-manifold \( M \) into \( R^m(c) \) with arbitrary codimension, we have

\[
\delta^R_k \leq 2(k - 1)(H^2 + c).
\]

The equality sign of (5.3) holds identically if and only if one of the following two cases occurs:

\begin{enumerate}
  \item \( M \) is a minimal Einstein submanifold such that, with respect to some suitable orthonormal frame \( \{e_1, \ldots, e_{2k}, e_{2k+1}, \ldots, e_m\} \), we have
    \[
    A_r = \begin{pmatrix}
      A_1^r & 0 \\
      \vdots & \ddots \\
      0 & A_k^r
    \end{pmatrix}, \quad r = 2k + 1, \ldots, m,
    \]
    where \( A_j^r, j = 1, \ldots, k, \) are symmetric \( 2 \times 2 \) submatrices satisfying \( \text{trace}(A_1^r) = \cdots = \text{trace}(A_k^r) = 0 \).
  \item \( M \) is a pseudo-umbilical Einstein submanifold such that, with respect to some suitable orthonormal frame \( \{e_1, \ldots, e_{2k}, e_{2k+1}, \ldots, e_m\} \), we have
    \[
    A_r = \begin{pmatrix}
      A_1^r & 0 \\
      \vdots & \ddots \\
      0 & A_k^r
    \end{pmatrix}, \quad r = 2k + 2, \ldots, m,
    \]
    where \( A_j^r, j = 1, \ldots, k, \) are symmetric \( 2 \times 2 \) submatrices satisfying \( \text{trace}(A_1^r) = \cdots = \text{trace}(A_k^r) = 0 \).
\end{enumerate}
Remark 12. The following example shows that inequality (5.3) does not hold for arbitrary submanifolds in general.

Example 1. Consider the following spherical hypercylinder: $M := S^2(1) \times E^{2k-2} \subset E^{2k+1}$. We have $\delta_k^{\text{Ric}} = 1$ and $H^2 = 1/k^2$ on $M$ which imply that $\delta_k^{\text{Ric}} = 1 > 2(k-1)/k^2 = 2(k-1)H^2$ for $k \geq 2$.

Theorem 7. Let $\phi : M \to \mathbb{R}^m(c)$ be an isometric immersion of an Einstein $n$-manifold $M$ into $\mathbb{R}^m(c)$. Then, for every natural number $q < n/2$, we have

$$\delta_q^{\text{Ric}} \leq \frac{n(n - q - 1)}{n - q} H^2 + (n - 1) - \frac{2q}{n} c.$$ (5.4)

The equality sign of (5.4) holds identically if and only if $M$ is totally geodesic.

Remark 13. The next example shows that inequality (5.4) does not hold for arbitrary submanifolds in general as well.

Example 2. For the spherical hypercylinder: $S^{n-q}(1) \times E^q \subset E^{n+1}$, we have $\delta_q^{\text{Ric}} = n - q - 1, \quad H^2 = \frac{(n-q)^2}{n^2}$ for $q < n/2$ which imply that $\delta_q^{\text{Ric}} > (n(n - q - 1)/(n - q))H^2$.

Some immediate consequences of Theorem 6 and Theorem 7 are the following.

Corollary 9. If a Riemannian manifold $M$ admits an isometric immersion into a Euclidean space which satisfies

$$\delta_q^{\text{Ric}} > \frac{n(n - q - 1)}{n - q} H^2, \quad n = \dim M,$$ (5.5)

for some natural number $q \leq n/2$ at some points, then $M$ is not Einstein.

This corollary applies to a large family of Riemannian manifolds. For instance, Example 1 and Corollary 9 imply immediate that $S^2 \times E^{2k-2}$ is not Einstein.

Theorem 6 and Theorem 7 also imply the following.

Corollary 10. If an Einstein $n$-manifold satisfies

$$\delta_q^{\text{Ric}} > (n - 1 - \frac{2q}{n})c$$ (5.6)
NASH’S EMBEDDING THEOREM

for some natural number \( q \leq n/2 \) at some points, then it admits no minimal isometric immersion into \( R^m(c) \) regardless of codimension.

**Corollary 11.** Let \( M \) be a compact Einstein \( n \)-manifold with finite fundamental group \( \pi_1 \) or with null first betti number, i.e., \( b_1 = 0 \). If there is a natural number \( q \leq n/2 \) such that \( \delta_{q}^{\text{Ric}} > 0 \), then \( M \) admits no Lagrangian isometric immersion into any complex \( n \)-torus or complex Euclidean \( n \)-space.

### 6. Solutions for the Family of Conformally Flat Spaces

In order to provide some solutions to Problem 3 for the special family of conformally flat manifolds endowed with the conformal flat structure, we need to define some special invariants as follows:

Let \( \text{Ric} \) denote the Ricci tensor of a Riemannian \( n \)-manifold \( M \). For each \( \ell \)-subspace \( L \) of \( T_p(M) \), \( p \in M \), we define the Ricci curvature \( S(L) \) of \( L \) as the trace of the restriction of the Ricci tensor \( \text{Ric} \) on \( L \), i.e.,

\[
S(L) = \text{Ric}(e_1, e_1) + \cdots + \text{Ric}(e_\ell, e_\ell)
\]  

for an orthonormal basis \( \{e_1, \ldots, e_\ell\} \) of \( L \). For each \( k \)-tuple \( (n_1, \ldots, n_k) \) in \( S(n) \), we introduce in [24] the Riemannian \( \sigma \)-invariant \( \sigma(n_1, \ldots, n_k) \) by

\[
\sigma(n_1, \ldots, n_k) = \tau - \inf \frac{(n-1) \sum_{j=1}^{k} (n_j - 1)}{(n-1)(n-2) + \sum_{j=1}^{k} n_j(n_j - 1)} S(L_j), \tag{6.2}
\]

where \( L_1, \ldots, L_k \) run over all \( k \) mutually orthogonal subspaces of \( T_pM \) such that \( \dim L_j = n_j, j = 1, \ldots, k \).

The following general optimal inequality from [24] for the family of conformally flat submanifolds provides another solution to Problem 3.

**Theorem 8.** Let \( M \) be a conformally flat \( n \)-manifold isometrically immersed in a Euclidean space. Then, for each \( k \)-tuple \( (n_1, \ldots, n_k) \) in \( S(n) \), we have

\[
\sigma(n_1, \ldots, n_k) \leq \alpha(n_1, \ldots, n_k) H^2, \tag{6.3}
\]

where \( \alpha(n_1, \ldots, n_k) \) are positive number defined by

\[
\alpha(n_1, \ldots, n_k) = \frac{n^2(n-1)(n-2)(n+k-1-\sum_{j=1}^{k} n_j)}{2((n-1)(n-2) + \sum_{j=1}^{k} n_j(n_j - 1))(n+k-\sum_{j=1}^{k} n_j)}. 
\]
The equality case of inequality (6.3) holds at a point \( p \in M \) if and only if, there exists an orthonormal basis \( e_1, \ldots, e_m \) at \( p \), such that the shape operators of \( M \) in \( R^n(\varepsilon) \) at \( p \) take the following forms:

\[
A_r = \begin{pmatrix}
A_r^1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & A_r^k \\
0 & \cdots & \mu_r I
\end{pmatrix}, \quad r = n+1, \ldots, m,
\]

(6.4)

where \( I \) is an identity matrix and \( A_j^r \)'s are symmetric \( n_j \times n_j \) submatrices satisfying

\[
\text{trace} (A_1^r) = \cdots = \text{trace} (A_k^r) = \mu_r.
\]

(6.5)

**Remark 14.** Inequality (6.3) does not hold for arbitrary submanifolds in general. This can be seen from the following example.

**Example 3.** Let \( M = S^{n-2}(1) \times \mathbb{E}^2 \subset \mathbb{E}^{n+1} = \mathbb{E}^{n-1} \times \mathbb{E}^2 \) denote the standard isometrically embedding of \( S^{n-2}(1) \times \mathbb{E}^2 \) in \( \mathbb{E}^{n+1} \) as a spherical hypercylinder over the unit \( (n-2) \)-sphere. If we choose \( k = 1, n_1 = 2 \), then we have \( \inf S(\pi) = 0 \), where \( \pi \) runs over all 2-planes of \( T_qM \) at a given point \( q \in M \). Hence, by (6.2), we have

\[
\sigma(2) = \tau - \frac{n-1}{n^2 - 3n + 4} \inf S(\pi) = \frac{(n-2)(n-3)}{2}. \tag{6.6}
\]

On the other hand, we have

\[
\alpha(2) = \frac{n^2(n-2)^2}{2(2n^2 - 3n + 4)}, \quad H^2 = \frac{(n-2)^2}{n^2}. \tag{6.7}
\]

Hence, we obtain

\[
\frac{(n-2)(n-3)}{2} = \sigma(2) > \alpha(2)H^2 = \frac{(n-2)^4}{2(2n^2 - 3n + 4)}, \quad \text{for } n > 4,
\]

which shows that the inequality (6.3) does not hold for an arbitrary submanifold in a Euclidean space in general.

An immediate application of Theorem 8 is the following.
Corollary 12. If a Riemannian $n$-manifold $M$ admits an isometric immersion into a Euclidean space whose $\sigma$-invariant satisfies

$$\sigma(n_1, \ldots, n_k) > \alpha(n_1, \ldots, n_k)H^2$$

at some points in $M$ for some $(n_1, \ldots, n_k)$ in $S(n)$, then $M$ is not conformally flat.

For instance, by applying this corollary, we conclude from Example 3 that, for any $n \geq 4$, the Riemannian product $S^{n-2} \times E^2$ is not conformally flat. On the other hand, it is well-known that $S^{n-1} \times \mathbb{R}$ is a conformally flat space.

Two other immediate consequences of Theorem 8 are the following obstructions in terms of $\sigma$-invariants to minimal and Lagrangian immersions.

Corollary 13. Let $M$ be a conformally flat $n$-manifold. If there exist a $k$-tuple $(n_1, \ldots, n_k)$ in $S(n)$ such that $\sigma(n_1, \ldots, n_k) > 0$ at some points in $M$, then $M$ does not admit any minimal isometric immersion into a Euclidean space.

Corollary 14. Suppose that $M$ is a compact conformally flat $n$-manifold either with finite fundamental group $\pi_1$ or with null first betti number $b_1$. If there exists a $k$-tuple $(n_1, \ldots, n_k)$ in $S(n)$ such that the $\sigma$-invariant $\sigma(n_1, \ldots, n_k) > 0$ at some points in $M$, then $M$ admits no Lagrangian isometric immersion into any complex $n$-torus or into the complex Euclidean $n$-space.

Remark 15. The condition on the $\sigma$-invariant $\sigma(n_1, \ldots, n_k)$ given in Corollary 14 is sharp. This is illustrated by the following example.

Example 4. Consider the Whitney immersion $w_a : S^n \rightarrow \mathbb{C}^n$ defined by

$$w_a(y_0, y_1, \ldots, y_n) = a(1 + i y_0)(y_1, \ldots, y_n), \quad a > 0,$$

with $y_0^2 + y_1^2 + \cdots + y_n^2 = 1$.

The Whitney $n$-sphere $W^n_a$ is the topological $n$-sphere $S^n$ endowed with the induced metric via (6.8). The Whitney $n$-sphere is a conformally flat space and the Whitney immersion is a Lagrangian immersion which has a unique self-intersection point at $w_a(-1, 0, \ldots, 0) = w_a(1, 0, \ldots, 0)$.

For any $k$-tuple $(n_1, \ldots, n_k)$ in $S(n)$, we have $\sigma(n_1, \ldots, n_k) \geq 0$ on $W^n_a$ with respect to the induced metric. Moreover, $\sigma(n_1, \ldots, n_k) = 0$ holds only at the
unique point of self-intersection. From these one may conclude that the condition on the \( \sigma \)-invariant in Corollary 14 is sharp.

**Remark 16.** Let \( F : S^1 \to \mathbb{C} \) be the unit circle in the complex plane defined by \( F(s) = e^{is} \) and let \( \iota : S^{n-1} \to E^n \) \((n \geq 3)\) be the unit hypersphere in \( E^n \) centered at the origin. Denote by \( f : S^1 \times S^{n-1} \to \mathbb{C}^n \) the complex extensor defined by \( f(s,p) = F(s) \otimes \iota(p), \, p \in S^{n-1} \). Then \( f \) is an isometric Lagrangian immersion of the conformally flat space \( M = S^1 \times S^{n-1} \) into \( \mathbb{C}^n \) which carries each pair \( \{(u,p), (-u,-p)\} \) of points in \( S^1 \times S^{n-1} \) to a point in \( \mathbb{C}^n \). Clearly, we have \( \pi_1(M) = \mathbb{Z} \) and \( b_1(M) = 1 \). Moreover, for each \( k \)-tuple \((n_1, \ldots, n_k) \in S(n)\), the \( \sigma \)-invariant \( \sigma(n_1, \ldots, n_k) \) on \( M \) is a positive constant.

This example illustrates that both conditions on \( \pi_1(M) \) and \( b_1(M) \) in Corollary 14 are necessary.

### 7. Solutions for Contact Metric Manifolds

By a contact manifold we mean a smooth \((2n+1)\)-manifold \( M \) together with a global 1-form \( \eta \) satisfying \( \eta \wedge (d\eta)^n \neq 0 \) on \( M \). An almost contact manifold \((M, \eta)\) is called an almost contact metric manifold if it admits an endomorphism \( \phi \) of its tangent bundle \( TM \), a vector field \( \xi \), and a Riemannian metric \( g \) such that

\[
\begin{align*}
\phi^2 &= -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi \xi = 0, \quad \eta \circ \phi = 0, \\
g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi),
\end{align*}
\]

for vector fields \( X, Y \) tangent to \( M \), where \( I \) is the identity endomorphism.

If \( \eta \) of an almost contact metric manifold \((M, \phi, \xi, \eta, g)\) is a contact form and if \( \eta \) satisfies \( d\eta(X, Y) = g(X, \phi Y) \) for all vectors \( X, Y \) tangent to \( M \), then \( M \) is called a contact metric manifold.

A contact metric manifold is called \( K \)-contact if its characteristic vector field \( \xi \) is a Killing vector field. It is well-known that the integral curves of \( \xi \) on a contact metric manifold are geodesics, i.e., we have \( \nabla_\xi \xi = 0 \). Moreover, a \( K \)-contact metric \((2n+1)\)-manifold satisfies

\[
\nabla_X \xi = -\phi X, \quad K(X \wedge \xi) = 1, \quad \text{Ric} (\xi) = 2n
\]

(7.2)
for $X \in \ker \eta$, where $K$ denotes the sectional curvature on $M$. A $K$-contact manifold is called Sasakian if we have $N_\phi + 2d\eta \otimes \xi = 0$, where $N_\phi$ is the Nijenhuis tensor associated to $\phi$.

On a Sasakian manifold $M$, one has

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X, \quad (7.3)$$

where $X,Y$ tangent to $M$. One may define an operator $h$ on an almost contact metric manifold $(M, \phi, \xi, \eta, g)$ by $h = \frac{1}{2}L_\xi \phi$. When the almost contact metric structure is a contact metric, $h$ is a symmetric operator satisfying

$$\nabla_X \xi = -\phi X - \phi hX, \quad h \circ \phi = -\phi \circ h, \quad \text{trace } h = 0, \quad \text{trace } h^2 = 2n - \text{Ric}(\xi), \quad (7.4)$$

where $\text{Ric}(\xi)$ is the Ricci curvature at $\xi$.

A submanifold $N$ of an almost contact metric manifold $(M, \phi, \xi, \eta, g)$ is called an invariant submanifold if the characteristic vector field $\xi$ is tangent to $N$ and, for each $X \in TN$, $\phi X$ is tangent to $N$.

Let $(M, \phi, \xi, \eta, g)$ be an almost contact metric $(2n+1)$-manifold. Assume that $f : M \to R^m(c)$ is an isometric immersion of $M$ into an $m$-dimensional real space form $R^m(c)$ of constant curvature $c$ with second fundamental form $\sigma$. For any natural number $k \in [2, 2n]$, we define the contact Riemannian invariant $\delta^c(k)$ by

$$\delta^c(k)(p) = \tau(p) - \inf_{L^k_\xi} \tau(L^k_\xi), \quad (7.5)$$

where $L^k_\xi$ runs over all linear $k$-subspace of $T_xM$ containing $\xi$.

For the invariant $\delta^c(k)$ we have the following results from [32].

**Theorem 9.** Let $(M, \phi, \xi, \eta, g)$ be an almost contact metric $(2n+1)$-manifold for which $\eta$ is a contact structure. Then, for any integer $k \in [2, 2n]$ and any isometric immersion of $M$ into a real space form $R^m(c)$, we have:

$$\delta^c(k) \leq \frac{(2n+1)^2(2n-k+1)}{2(2n-k+2)} \|H\|^2 + \frac{1}{2}(2n(2n+1) - k(k-1))c. \quad (7.6)$$

Moreover, the equality case holds identically if and only if we have:
(a) \( n + 1 \leq k \leq 2n \) and
(b) with respect to some suitable orthonormal basis \( \{e_1, \ldots, e_{2n+1}, e_{2n+2}, \ldots, e_m\} \)
with \( \xi = e_1 \), the shape operator of \( M \) takes the following form:

\[
A_r = \begin{pmatrix}
A_{k+1}^r & 0 \\
0 & \mu_r I
\end{pmatrix}, \quad r \in \{2n+2, \ldots, m\},
\]

(7.7)

where \( A_{k+1}^r \) are symmetric \((k+1)\times(k+1)\) submatrices satisfying \( \text{trace } A_{k+1}^r = \mu_r \) for \( r = 2n+2, \ldots, m \).

**Theorem 10.** If \( f : M \to \mathbb{R}^m(c) \) is an isometric immersion of a K-contact \((2n+1)\)-manifold \( M \) into a real space form \( \mathbb{R}^m(c) \) which satisfies the equality case of (7.6) for some integer \( k \in [n+1, 2n] \), then we have \( c \geq 1 \). In particular, when \( c = 1 \), the K-contact structure on \( M \) is Sasakian.

**Theorem 11.** If a K-contact \((2n+1)\)-manifold \( M \) admits an isometric immersion into an \( m \)-sphere \( S^m(c) \) of constant curvature \( c \) which satisfies the equality case of (7.6) with \( k = 2 \) identically, then we have

(1) \( c = n = 1 \).
(2) \( M \) is a Sasakian manifold of constant curvature one.
(3) The immersion is totally geodesic.

**Theorem 12.** If a K-contact \((2n+1)\)-manifold \( M \) admits an isometric immersion into \( S^m(1) \) which satisfies the equality case of (7.6) with \( k = n + 1 \), then \( M \) is a minimal submanifold of \( S^m(1) \).

**Theorem 13.** If a K-contact \((2n+1)\)-manifold \( M \) admits an isometric immersion into \( S^m(1) \) which satisfies the equality case of (7.6) with \( k = 3 \), then \( n = 2 \) and we have either

(a) \( M \) is a Sasakian manifold of constant curvature one isometrically immersed in \( S^m(1) \) as a totally geodesic submanifold, or
(b) \( M \) is a Sasakian 5-manifold foliated by Sasakian 3-manifolds of constant curvature one and \( M \) is isometrically immersed in \( S^m(1) \) as a minimal submanifold of codimension at least two. Moreover, leaves of the foliation are immersed as totally geodesic submanifolds of \( S^m(1) \).
8. Contact Number of a Euclidean Submanifold

In Sections 8–10, we present another solution to Problem 1. In fact, we will introduce an invariant, called the contact number, associated with each arbitrary Euclidean submanifold of dimension $\geq 2$. Surprisingly, this invariant relates closely with the well-known notion of isotropic submanifolds in the sense of O'Neill [68]. The notion of contact number also relates to the notion of holomorphic curves. In section 8 we will also present a simple and useful criterion for a submanifold to have any given contact number (For the details, see [26, 31]).

Let $M$ be an $n$-dimensional submanifold in a Euclidean space $\mathbb{E}^m$ with arbitrary codimension. For any given point $p \in M$ and given unit vector $u$ in the unit tangent bundle $U_p M$, there exists a unique unit speed geodesic $\gamma_u$ in $M$ through $p$ satisfying $\gamma_u(0) = p$ and $\gamma_u'(0) = u$.

For the same pair $(p,u)$, there is another canonical unit speed curve $\beta_u$ associated with $(p,u)$ which is called the normal section defined as follows:

Let $E(p,u)$ be the affine $(m - n + 1)$-subspace of $\mathbb{E}^m$ through $p$ spanned by $u$ and the normal space $T_p^\perp M$ at $p$. The intersection of $M$ and $E(p,u)$ gives rise to a unit speed curve $\beta_u(s)$ with $\beta_u(0) = p$ and $\beta_u'(0) = u$ defined on an open interval containing 0. This curve $\beta_u$ is called the normal section at $(p,u)$.

The notion of normal sections for surfaces in Euclidean 3-space have been used since L. Euler in 1760. On the other hand, for arbitrary Euclidean submanifolds with arbitrary codimension, this notion of normal sections have been investigated in [9] from a different angle. Since then normal sections have been revisited by many geometers (see, for instance, [9, 31, 34, 40, 41, 42, 45, 46, 55, 63, 79]). In particular, C. U. Sánchez et al. showed that normal sections play some important roles in algebraic geometry as well as in differential geometry.

The geodesic $\gamma_u$ and the normal section $\beta_u$ at $(p,u)$ are said to be in contact of order $k$ if $\gamma_u^{(i)}(0) = \beta_u^{(i)}(0)$ for $i = 1, \ldots, k$, where $\gamma_u^{(i)}$ and $\beta_u^{(i)}$ denote the $i$-th derivatives of $\gamma_u$ and $\beta_u$ with respect to their arclength functions.

The notion of contact number was introduced in [31] as follows.

**Definition 2.** A Euclidean submanifold $M$ of dimension $\geq 2$ is said to be in contact of order $k$ if, for each $p \in M$ and $u \in U_p M$, the geodesic $\gamma_u$ and the normal section $\beta_u$ at $(p,u)$ are in contact of order $k$. If the Euclidean submanifold
$M$ is in contact of order $k$ for every natural number $k$, the contact number $c_#(M)$ of $M$ is defined to be $\infty$. Otherwise, the contact number $c_#(M)$ is defined to be the largest natural number $k$ such that $M$ is in contact of order $k$ and but not of order $k + 1$.

**Example 5.** The contact number of an affine $n$-plane of a Euclidean space is $\infty$. And the contact number of a hypersphere of a Euclidean space is $\infty$.

**Example 6.** The contact number of a circular cylinder in $\mathbb{E}^3$ is 2.

**Example 7.** Let $M$ be a compact Riemannian manifold. Then $M$ has a unique kernel of the heat equation: $K : M \times M \times \mathbb{R}_0^+ \to \mathbb{R}$. Let $\delta$ denote the distance function on $M$. Then $M$ is called a strongly harmonic manifold if there exists a function $\Psi : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ such that $K(x, y, t) = \Psi(\delta(x, y), t)$ for $x, y \in M$ and $t \in \mathbb{R}_0^+$. Compact symmetric spaces of rank one are known examples of strongly harmonic manifolds ([1, p.158]).

Let $\lambda_k$ be the $k$-th nonzero eigenvalue of the Laplacian $\Delta$. Denote by $V_k$ be the eigenspace of $\Delta$ with eigenvalue $\lambda_k$. On $V_k$ we define an inner product by $\langle f, g \rangle = \int_M fg$ for $f, g \in V_k$. $V_k$ together with $\langle \cdot, \cdot \rangle$ is a finite dimensional Euclidean space. Let $\varphi_k^1, \ldots, \varphi_k^m$ be an orthonormal basis of $V_k$. Then the mapping

$$\varphi_k : M \to \mathbb{E}^m : x \mapsto c_k(\varphi_k^1(x), \ldots, \varphi_k^m(x))$$

defines a helical isometric immersion for some suitable constant $c_k$. Such submanifolds satisfy $c_#(M) = \infty$. In particular, if $M = S^2(\sqrt{3}/a)$ denotes the 2-sphere with constant sectional curvature $3/a^2$, then $\varphi_2^2 : S^2(\sqrt{3}/a) \to S^4(1/a) \subset \mathbb{E}^5$ is given by

$$\varphi_2^2 = a\left(\frac{1}{\sqrt{3}}yz, \frac{1}{\sqrt{3}}xz, \frac{1}{\sqrt{3}}xy, \frac{1}{2\sqrt{3}}(x^2 - y^2), \frac{1}{6}(x^2 + y^2 - 2z^2)\right), \quad (8.1)$$

where $x^2 + y^2 + z^2 = 3$. This isometric immersion is called a Veronese surface.

**Example 8.** A Euclidean submanifold $M$ is said to have geodesic normal sections if, for each point $x \in M$ and each unit tangent vector $u \in T_xM$, the corresponding normal section $\beta_u$ and geodesic $\gamma_u$ coincide (see [34]).

Let $\psi_j : M \to \mathbb{E}^{m_j}$ ($j = 1, \ldots, s$) be $s$ isometric immersions of a Riemannian manifold with geodesic normal sections. For real numbers $c_1, \ldots, c_s$ with $c_1^2 + \ldots + c_s^2 = a^2$, Let $\psi : M \to \mathbb{E}^m$ be the isometric immersion with $\psi = \sum_{j=1}^s c_j \psi_j$.
\[ \cdots + c_s^2 = 1, \text{ the diagonal immersion: } (c_1 \psi_1, \ldots, c_s \psi_s) : M \to \mathbb{E}^{1+m_1+\cdots+m_s} \text{ defined by } p \mapsto (c_1 \psi_1(p), \ldots, c_s \psi_s(p)) \text{ satisfies } \epsilon_\#(M) = \infty. \]

**Example 9.** Consider the flat surface \( M \) in \( \mathbb{E}^6 \) given by a flat torus \( T_{3,a} \) whose immersion into \( \mathbb{E}^6 \) is congruent to

\[
\psi_3^a(u, v) = \frac{2}{\sqrt{6}a} \begin{pmatrix}
\frac{1}{\sqrt{2}} \cos \sqrt{2}au, & \frac{1}{\sqrt{2}} \sin \sqrt{2}au, & \cos \frac{au}{\sqrt{2}}, & \frac{\sqrt{3av}}{\sqrt{2}}, \\
\cos \frac{au}{\sqrt{2}}, & \sin \frac{\sqrt{3av}}{\sqrt{2}}, & \sin \frac{au}{\sqrt{2}}, & \frac{\sqrt{3av}}{\sqrt{2}}.
\end{pmatrix}
\]

(8.2)

It can be shown that the contact number of this flat torus in \( \mathbb{E}^6 \) is equal to 4.

The next result from [26] provides many examples of flat tori in Euclidean spaces with high contact numbers.

**Theorem 14.** (1) For any positive number \( a \) and any integer \( n \geq 2 \), the map

\[
\psi_{2n}^a(x, y) = \frac{1}{\sqrt{2n}a} \left( \cos \left( \sqrt{1 + \cos \left( \frac{(2j - 1)\pi}{2n} \right)ax + \sqrt{1 - \cos \left( \frac{(2j - 1)\pi}{2n} \right)ay} \right), \\
\sin \left( \sqrt{1 + \cos \left( \frac{(2j - 1)\pi}{2n} \right)ax + \sqrt{1 - \cos \left( \frac{(2j - 1)\pi}{2n} \right)ay} \right), \\
\cos \left( \sqrt{1 + \cos \left( \frac{(2j - 1)\pi}{2n} \right)ax - \sqrt{1 - \cos \left( \frac{(2j - 1)\pi}{2n} \right)ay} \right), \\
\sin \left( \sqrt{1 + \cos \left( \frac{(2j - 1)\pi}{2n} \right)ax - \sqrt{1 - \cos \left( \frac{(2j - 1)\pi}{2n} \right)ay} \right) \right)_{j=1,\ldots,n}
\]

defines an isometric immersion of a flat 2-torus \( T_{2n,a} \) into \( \mathbb{E}^{4n} \) with contact number \( 4n - 2 \). Moreover, the immersion has parallel mean curvature vector.

(2) For any positive number \( a \) and any integer \( n \geq 2 \), the map

\[
\psi_{2n+1}^a(x, y) = \frac{1}{\sqrt{2n+1}a} \left( \cos \left( \sqrt{2ax}, \sin \left( \sqrt{2ax} \right), \\
\cos \left( \sqrt{1 + \cos \left( \frac{2j\pi}{2n+1} \right)ax + \sqrt{1 - \cos \left( \frac{2j\pi}{2n+1} \right)ay} \right), \\
\right)_{j=1,\ldots,n}
\]

defines an isometric immersion of a flat 2-torus $T_{2n+1,a}$ into $\mathbb{E}^{4n+2}$ with contact number $4n$. Moreover, the immersion has parallel mean curvature vector.

(3) Conversely, if $M$ is a flat surface in $\mathbb{E}^{2k}$ ($k \geq 3$) with contact number $c_\#(M) \geq 2k - 2$ and with parallel mean curvature vector, then $M$ is either an open portion of a totally geodesic 2-plane, or an open portion of the flat torus $T_{3,a}$ for some $a > 0$ defined in Example 9 whose immersion is congruent to the $\psi_3^a$ defined by (8.2), or an open portion of the flat torus $T_{k,a}$ for some $a > 0$ defined above whose immersion is congruent to the $\psi_k^a$ defined by either by (8.3) or by (8.4), according to $k = 2n \geq 4$ or $k = 2n + 1 \geq 5$.

Remark 17. Theorem 14 implies that, for any even number $2k \geq 6$, there are some flat tori in Euclidean space whose contact number is $2k$, no matter how large $k$ is.

9. Relations Between Contact Number, Isotropy, Holomorphic Curves and Special Lagrangian Surfaces

The following result from [31] gives a very simple relationship between the contact number of a Euclidean submanifold and the notion of isotropic submanifolds.

**Theorem 15.** For every Euclidean submanifold $M$, we have:

1. The contact number $c_\#(M)$ of $M$ is at least 2, i.e., $c_\#(M) \geq 2$.
2. $M$ is isotropic if and only if $c_\#(M) \geq 3$ holds.
3. $M$ is constant isotropic if and only if $c_\#(M) \geq 4$ holds.

The next result from [31] classifies completely Euclidean submanifolds of codimension two with contact number $\geq 3$. 

Theorem 16. Let $M$ be an $n$-dimensional submanifold of $\mathbb{E}^{n+2}$. Then $c_{\#}(M) \geq 3$ holds if and only if one of the following three cases occurs:

1. $c_{\#}(M) = 3$, $n = 2$, and $M$ is a complex curve lying linearly fully in $\mathbb{C}^2$, where $\mathbb{C}^2$ denotes $\mathbb{E}^4$ endowed with some orthogonal complex structure.
2. $c_{\#}(M) = \infty$ and $M$ is an open portion of an $n$-plane.
3. $c_{\#}(M) = \infty$ and $M$ is an open portion of a hypersphere lying in a hyperplane of $\mathbb{E}^{n+2}$.

In particular, for hypersurfaces we have the following.

Corollary 15. Let $M$ be a hypersurface of Euclidean $(n+1)$-space $\mathbb{E}^{n+1}$. Then one of the following three cases must occur:

1. $c_{\#}(M) = 2$.
2. $c_{\#}(M) = \infty$ and $M$ is an open portion of a hyperplane.
3. $c_{\#}(M) = \infty$ and $M$ is an open portion of a hypersphere.

The following theorem from [31] provides a very simple characterization of holomorphic curves in $\mathbb{C}^2$.

Theorem 17. A surface $M$ in $\mathbb{E}^4$ satisfies $c_{\#}(M) = 3$ if and only if it is a non-planar holomorphic curve with respect to some orthogonal complex structure on $\mathbb{E}^4$.

Combining Theorem 17 and a result of [33] yields the following simple characterization of special Lagrangian surfaces in $\mathbb{C}^2$.

Corollary 16. Every non-planar minimal Lagrangian surface $M$ in the complex Euclidean plane $\mathbb{C}^2$ satisfies $c_{\#}(M) = 3$.

The next theorem gives a simple and very useful criterion for a Euclidean submanifold to have any given contact number.

Theorem 18. A submanifold in a Euclidean space is in contact of order $k$ ($k \geq 3$) if and only if each $u \in UM$ is an eigenvector of $A(\nabla^j h)(\omega^{j+2})$ for $j = 0, \ldots, k-3$.

An immediate application of Theorem 18 is the following.
Corollary 17. Every isotropic submanifold with parallel second fundamental form in a Euclidean space satisfies $c_\#(M) = \infty$.

Remark 18. Not every Euclidean submanifold $M$ with parallel second fundamental form satisfies $c_\#(M) = \infty$. For instance, a circular cylinder $\mathbb{R} \times S^1$ in $E^3$ has parallel second fundamental form, but its contact number is 2, not $\infty$.

Remark 19. For the details concerning contact number, see [26, 31].

10. First Non-Trivial Examples of Pseudo-Umbilical Submanifolds

The following result classify surfaces of $c_\#(M) \geq 4$ in $E^6$ either with constant mean curvature or with constant Gauss curvature.

Theorem 19. Let $M$ be a surface in $E^6$ with constant mean curvature or constant Gauss curvature. If $c_\#(M) \geq 4$, then either $c_\#(M) = \infty$ or $c_\#(M) = 4$ holds. Moreover, we have:

(1) If $c_\#(M) = \infty$, $M$ is one of the following three surfaces:
   (a) An open portion of a 2-plane.
   (b) An open portion of an ordinary 2-sphere lying in a 3-plane of $E^6$.
   (c) An open portion of the Veronese surface $S^2(\sqrt{3}/a)$ contained in $S^4(1/a)$ which lies in a hyperplane of $E^6$.

(2) If $c_\#(M) = 4$, then $M$ is one of the following two surfaces:
   (a) $M$ is an open portion of a flat torus $T^2_a$ for some $a > 0$ and its immersion is congruent to
      \[
      \varphi(u, v) = \frac{2}{\sqrt{6}a}\left(\cos \frac{au}{\sqrt{2}} \cos \frac{\sqrt{3}av}{\sqrt{2}}, \cos \frac{au}{\sqrt{2}} \sin \frac{\sqrt{3}av}{\sqrt{2}}; \frac{1}{\sqrt{2}} \cos(\sqrt{2}au), \right.
      \sin \frac{au}{\sqrt{2}} \cos \frac{\sqrt{3}av}{\sqrt{2}}, \sin \frac{au}{\sqrt{2}} \sin \frac{\sqrt{3}av}{\sqrt{2}}; \frac{1}{\sqrt{2}} \sin(\sqrt{2}au)\). \tag{10.1}
   
   (b) $M$ is an open portion of $S^2(\sqrt{3}/a)$ immersed linearly fully in $E^6$ as a pseudo-umbilical surface with non-parallel mean curvature vector.

The following result classifies surfaces of case (2-b) in Theorem 19.
**Theorem 20.** Let \( a > 0 \) and let \( \{ \lambda(u, v), \mu(u, v) \} \) be non-trivial solutions of the system of partial differential equations:

\[
\lambda_v = \left( \mu \cos \left( \frac{au}{\sqrt{3}} \right) \right)_u, \quad (10.2)
\]

\[
\lambda^2 \mu_v + \cos \left( \frac{au}{\sqrt{3}} \right) \lambda_u \mu^2 - \frac{2a}{\sqrt{3}} \sin \left( \frac{au}{\sqrt{3}} \right) \lambda\mu^2 - 2\lambda\lambda_v \mu = \frac{a}{\sqrt{3}} \sin \left( \frac{au}{\sqrt{3}} \right) \lambda^3, \quad (10.3)
\]

defined on a simply-connected open set \( V_{\lambda, \mu}^a \). Let \( U_{\lambda, \mu}^a \) be the open subset of \( S^2(\sqrt{3}/a) \) with metric \( g = du^2 + \cos^2 \left( \frac{au}{\sqrt{3}} \right) dv^2 \) defined on \( V_{\lambda, \mu}^a \). Then, up to rigid motions, there exists a unique pseudo-umbilical isometric immersion \( \psi_{\lambda, \mu}^a : U_{\lambda, \mu}^a \to \mathbb{E}^6 \) with contact number 4, constant mean curvature \( a \), and whose mean curvature vector satisfies

\[
D_{\partial/\partial u} H = a\lambda \xi, \quad D_{\partial/\partial v} H = a\mu \cos \left( \frac{au}{\sqrt{3}} \right) \xi, \quad (10.4)
\]

where \( \xi \) is a unit normal vector filed orthogonal to the first normal bundle of \( \psi_{\lambda, \mu}^a \).

Conversely, every surface in \( \mathbb{E}^6 \) with contact number 4, constant mean curvature, and non-parallel mean curvature vector is given by a pseudo-umbilical immersion of an open portion of \( S^2(\sqrt{3}/a) \) which is congruent to a \( \psi_{\lambda, \mu}^a \) obtained in the way described above.

After solving the partial differential equations (10.2) and (10.3), we obtain the first non-trivial examples of pseudo-umbilical surfaces in Euclidean space. Those pseudo-umbilical surfaces do not contained in any hypersphere of the Euclidean space.

**Example 10.** For any two positive numbers \( a, c \), we put

\[
\beta = \frac{2}{3}(a^2 + 6c^2), \quad \delta = \frac{2}{3} \sqrt{a^4 + 6a^2c^2 + 36c^4}, \quad (10.5)
\]

Consider the map \( \varphi \) defined by

\[
\varphi(u, v) = \cos^2 \left( \frac{au}{\sqrt{3}} \right) \left( \frac{\sqrt{3}}{a} \tan \left( \frac{au}{\sqrt{3}} \right) \cos \left( \frac{av}{\sqrt{3}} \right) \right)_u.
\]

\[
\frac{3\delta + 3\beta - 4a^2}{6\delta \sqrt{\beta - \delta}} \sin(\sqrt{\beta - \delta}v) + \frac{(3\delta - 3\beta + 4a^2)}{6\delta \sqrt{\beta + \delta}} \sin(\sqrt{\beta + \delta}v),
\]

\[
\frac{\sqrt{2ac} \sin(\sqrt{\beta - \delta}v)}{\delta \sqrt{\beta + \delta}} - \frac{\sqrt{2ac} \sin(\sqrt{\beta + \delta}v)}{\delta \sqrt{\beta - \delta}}.
\]
\[ \frac{(2a^2 - 3\beta - 3\delta) \cos(\sqrt{\beta - \delta})}{4a\delta} - \frac{(2a^2 - 3\beta + 3\delta) \cos(\sqrt{\beta + \delta})}{4a\delta}, \]
\[ \frac{(2a^2 + 3\beta + 3\delta) \cos(\sqrt{\beta - \delta})}{4\sqrt{3}a\delta} - \frac{(2a^2 + 3\beta - 3\delta) \cos(\sqrt{\beta + \delta})}{4\sqrt{3}a\delta}. \]

(10.6)

The map \( \varphi \) gives rise a pseudo-umbilical immersion of \( S^2(\sqrt{3}/a) \) in \( \mathbb{E}^6 \) with non-parallel mean curvature vector.

In particular, if we choose \( a\sqrt{3} \) and \( c = 1 \), the immersion (10.6) reduces to the following non-trivial pseudo-umbilical immersion of \( S^2(\sqrt{3}/a) \) in \( \mathbb{E}^6 \):

\[
\varphi(u, v) = \cos^2 u \left( \tan u \cos v, \tan u \sin v, \right.
\]
\[
\frac{3 + \sqrt{7}}{2\sqrt{70} - 14\sqrt{7}} \sin(\sqrt{10 - 2\sqrt{7}} v) + \frac{(\sqrt{7} - 3)}{2\sqrt{70} + 14\sqrt{7}} \sin(\sqrt{10 + 2\sqrt{7}} v),
\]
\[
\frac{\sqrt{5} - \sqrt{7}}{2\sqrt{42}} \sin(\sqrt{10 - 2\sqrt{7}} v) - \frac{\sqrt{5} + \sqrt{7}}{2\sqrt{42}} \sin(\sqrt{10 + 2\sqrt{7}} v),
\]
\[
\frac{12 + 3\sqrt{7}}{4\sqrt{21}} \cos(\sqrt{10 - 2\sqrt{7}} v) + \frac{3\sqrt{7} - 12}{4\sqrt{21}} \cos(\sqrt{10 + 2\sqrt{7}} v),
\]
\[
\frac{7 + 6\sqrt{7}}{28} \cos(\sqrt{10 - 2\sqrt{7}} v) + \frac{7 - 6\sqrt{7}}{28} \cos(\sqrt{10 + 2\sqrt{7}} v) \right).
\]
(10.7)

11. Kählerian Version

Let \( M^n \) be a Kähler manifold of complex dimension \( n \). Denote by \( J \) the complex structure on Kähler manifold. For each plane section \( \pi \subset T_x M, x \in M \), we denote by \( K(\pi) \) the sectional curvature of the plane section \( \pi \) as before. A plane section \( \pi \subset T_x M \) is called \emph{totally real} if \( J\pi \) is perpendicular to \( \pi \). For each real number \( k \) we define a Kählerian invariant \( \delta_k^\pi \) by

\[
\delta_k^\pi(x) = \tau(x) - k \inf K^\pi(x), \quad x \in M, \tag{11.1}
\]

where \( \inf K^\pi(x) = \inf_{\pi''} \{ K(\pi'') \} \) and \( \pi'' \) runs over all totally real plane sections in \( T_x M \). This type of invariants is similar to the invariants defined in earlier sections of this article.
A Kähler manifold $\tilde{\mathcal{M}}^n(4c)$ of constant holomorphic sectional curvature $4c$ is called a complex space form. There are three types of complex space forms: elliptic, hyperbolic, or flat according as the holomorphic sectional curvature is positive, negative, or zero.

Let $\mathbb{CP}^m(4c)$ be a complex projective $m$-space endowed with the Fubini-Study metric of constant holomorphic sectional curvature $4c$. Then $\mathbb{CP}^m(4c)$ is a complete and simply-connected elliptic complex space form.

Complex Euclidean space $\mathbb{C}^m$ endowed with the usual Hermitian metric is a complete and simply-connected flat complex space form.

Let $D_m$ be the open unit ball in $\mathbb{C}^m$ endowed with the natural complex structure and the Bergman metric of constant holomorphic sectional curvature $4c$, $c < 0$. Then $D_m$ is a complete and simply-connected hyperbolic complex space form.

By a Kähler submanifold of a Kähler manifold we mean a complex submanifold with the induced Kähler structure. Just like real case, we denote by $h$ and $A$ the second fundamental form and the shape operator of $M^n$ in $\tilde{\mathcal{M}}^{n+p}$, respectively, for a Kähler submanifold $M^n$ of a Kähler manifold $\tilde{\mathcal{M}}^{n+p}$.

For the Kähler submanifold we consider an orthonormal frame
e_1, \ldots, e_n, e_1^* = Je_1, \ldots, e_n^* = Je_n
of the tangent bundle and an orthonormal frame\[\xi_1, \ldots, \xi_p, \xi_1^* = J\xi_1, \ldots, \xi_p^* = J\xi_p\]of the normal bundle. With respect to such an orthonormal frame, the complex structure $J$ on $M$ is given by\[
J = \begin{pmatrix}
0 & -I_n \\
I_n & 0
\end{pmatrix},
\]where $I_n$ denotes an identity matrix of degree $n$.

For a Kähler submanifold $M^n$ in $\tilde{\mathcal{M}}^{n+p}$ the shape operator of $M^n$ satisfies (see, for instance, [66])\[
A_J\xi_r = JA_r, \quad JA_r = -A_rJ, \quad \text{for } r = 1, \ldots, n, 1^*, \ldots, p^*,
\]
where \( A_r = A_{\xi_r} \). From these it follows that the shape operator of \( M^n \) takes the form:

\[
A_\alpha = \begin{pmatrix}
A'_\alpha & A''_\alpha \\
A''_\alpha & -A'_\alpha
\end{pmatrix}, \quad A_{\alpha^*} = \begin{pmatrix}
-A''_\alpha & A'_\alpha \\
A'_\alpha & A''_\alpha
\end{pmatrix}, \quad \alpha = 1, \ldots, p,
\]

(11.4)

where \( A'_\alpha \) and \( A''_\alpha \) are \( n \times n \) matrices. This condition implies that every Kähler submanifold \( M^n \) is minimal, i.e.,

\[
\text{trace } A_\alpha = \text{trace } A_{\alpha^*} = 0, \quad \alpha = 1, \ldots, p.
\]

The following notion of strongly minimal Kähler submanifolds was first introduced in [19].

**Definition 3.** A Kähler submanifold \( M^n \) of a Kähler manifold \( \tilde{M}^{n+p} \) is called **strongly minimal** if it satisfies

\[
\text{trace } A'_\alpha = \text{trace } A''_\alpha = 0, \quad \text{for } \alpha = 1, \ldots, p,
\]

(11.5)

with respect to some orthonormal frame: \( e_1, \ldots, e_n, e_{1^*} = J e_1, \ldots, e_{n^*} = J e_n \), \( \xi_1, \ldots, \xi_p, \xi_{1^*} = J \xi_1, \ldots, \xi_{p^*} = J \xi_p \).

For Kähler submanifolds in complex space forms, we have the following sharp general result from [19] which can be regarded as a Kählerian version of Theorem 1.

**Theorem 21.** For any Kähler submanifold \( M^n \) of complex dimension \( n \geq 2 \) in a complex space form \( \tilde{M}^{n+p}(4c) \), the following statements hold.

1. For each \( k \in (-\infty, 4] \), \( \delta^*_k \) satisfies

\[
\delta^*_k \leq (2n^2 + 2n - k)c.
\]

(11.6)

2. Inequality (1.7) fails for every \( k > 4 \).

3. \( \delta^*_k = (2n^2 + 2n - k)c \) holds identically for some \( k \in (-\infty, 4) \) if and only if \( M^n \) is a totally geodesic Kähler submanifold of \( \tilde{M}^{n+p}(4c) \).

4. The Kähler submanifold \( M^n \) satisfies \( \delta^*_4 = (2n^2 + 2n - 4)c \) at a point \( x \in M^n \) if and only if there exists an orthonormal basis

\[
e_1, \ldots, e_n, e_{1^*} = J e_1, \ldots, e_{n^*} = J e_n, \xi_1, \ldots, \xi_p, \xi_{1^*} = J \xi_1, \ldots, \xi_{p^*} = J \xi_p
\]

(11.7)
of $T_x\tilde{M}^{n+p}(4c)$ such that, with respect to this basis, the shape operator of $M^n$ takes the following form:

$$A_\alpha = \begin{pmatrix} A'_\alpha & A''_\alpha \\ A''_\alpha & -A'_\alpha \end{pmatrix}, \quad A_\alpha^* = \begin{pmatrix} -A''_\alpha & A'_\alpha \\ A'_\alpha & A''_\alpha \end{pmatrix},$$

$$A'_\alpha = \begin{pmatrix} a_\alpha & b_\alpha \\ b_\alpha & -a_\alpha \end{pmatrix}, \quad A''_\alpha = \begin{pmatrix} a'^*_\alpha & b'^*_\alpha \\ b'^*_\alpha & -a'^*_\alpha \end{pmatrix},$$

for some $n \times n$ matrices $A'_\alpha, A''_\alpha, \alpha = 1, \ldots, p$.  

**Theorem 22.** A complete Kähler submanifold $M^n (n \geq 2)$ in $CP^{n+p}(4c)$ satisfies

$$\delta^r_4 = 2(n^2 + n - 2)c$$

identically if and only if

(1) $M^n$ is a totally geodesic Kähler submanifold, or

(2) $n = 2$ and $M^2$ is a strongly minimal Kähler surface in $CP^{2+p}(c)$.

**Theorem 23.** A complete Kähler submanifold $M^n (n \geq 2)$ of $C^{n+p}$ satisfies

$$\delta^r_4 = 0$$

identically if and only if

(1) $M^n$ is a complex $n$-plane of $C^{n+p}$, or

(2) $M^n$ is a complex cylinder over a strongly minimal Kähler surface $M^2$ in $C^{n+p}$ (i.e., $M$ is the product submanifold of a strongly minimal Kähler surface $M^2$ in $C^{p+2}$ and the identity map of the complex Euclidean $(n-2)$-space $C^{n-2}$).

**Remark 20.** For a Kähler manifold $M$ of complex dimension $n$, one may extend the invariant $\delta^r_4$ to $\delta^r_{\ell,k}$ as

$$\delta^r_{\ell}(x) = \tau(x) - \frac{k}{\ell-1} \inf_{L^r_\ell} \tau(L^r_\ell), \quad x \in M,$$

where $L^r_\ell$ runs over all totally real $\ell$-subspaces of $T_xM$. 
For each integer \( \ell \in [2, n] \), inequality (11.6) was extended in [77] to the following inequality:

\[
\delta_{r,\ell,k}(x) \leq \left\{ 2n^2 + 2n - \frac{k}{4} \left( \frac{\ell}{2} \right) \right\} c
\]

(11.12)

for Kähler submanifolds in a complex space form \( \tilde{M}^m(4c) \). However, for \( \ell \geq 3 \), the equality sign of (11.12) occurs only for totally geodesic Kähler submanifolds.

**Remark 21.** Every totally geodesic Kähler submanifold of a complex space form is trivially strongly minimal. There also exist nontrivial examples of strongly minimal Kähler submanifolds.

**Example 11.** Consider the complex quadric \( Q_2 \) in \( CP^3(4c) \) defined by

\[
Q_2 = \left\{ (z_0, z_1, z_2, z_3) \in CP^3(4c) : z_0^2 + z_1^2 + z_2^2 + z_3^2 = 0 \right\},
\]

(11.13)

where \( \{z_0, z_1, z_2\} \) is a homogeneous coordinate system of \( CP^3(4c) \).

It is known that the scalar curvature \( \tau \) of \( Q_2 \) equals to \( 8c \) and \( \inf K^r = 0 \). Thus, we obtain \( \delta^r = 8c \). Thus, \( Q_2 \) is a non-totally geodesic Kähler submanifold which satisfies (1.10) with \( n = 2 \). Therefore, according to Theorem 21, \( Q_2 \) is a strongly minimal Kähler surface in \( CP^3(4c) \).

On the other hand, it is also well-known that \( Q_2 \) is an Einstein-Kähler surface with Ricci tensor \( S = 4cg \), where \( g \) is the metric tensor of \( Q_2 \). Thus, the equation of Gauss yields

\[
g(A_1^2 X, Y) = cg(X, Y), \quad X, Y \in TQ_2.
\]

(11.14)

Hence, with respect to a suitable choice of \( e_1, e_2, Je_1, Je_2, \xi_1, J\xi_1 \), we have

\[
A_1 = \begin{pmatrix} A_1' & A_1'' \\ A_1'' & -A_1' \end{pmatrix}, \quad A_1^* = \begin{pmatrix} -A_1'' & A_1' \\ A_1' & A_1'' \end{pmatrix},
\]

(11.15)

where

\[
A_1' = \begin{pmatrix} \sqrt{c} & 0 \\ 0 & -\sqrt{c} \end{pmatrix}, \quad A_1'' = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

(11.16)

This also shows that \( Q_2 \) is strongly minimal in \( CP^3(4c) \).
Another nontrivial example of strongly minimal Kähler surface in \( \mathbb{C}^3 \) is given by the following.

**Example 12.** The complex surface:
\[
\left\{ z \in \mathbb{C}^3 : z_1^2 + z_2^2 + z_3^2 = 1 \right\}, \tag{11.17}
\]
is a strongly minimal Kähler surface in \( \mathbb{C}^3 \).

The next examples of strongly minimal Kähler surface in \( \mathbb{C}^3 \) are given in [77].

**Example 13.** The complex surfaces:
\[
N^2_k = \left\{ z \in \mathbb{C}^3 : z_1 + z_2 + z_3^2 = k \right\}, \quad k \in \mathbb{C}, \tag{11.18}
\]
are strongly minimal Kähler surfaces in \( \mathbb{C}^3 \).

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**References**


