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THE GEOMETRY OF EYE AND BRAIN

BY

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Dedicated to the sixtieth birthday of Professor Bang-Yen Chen

0. Sketch of Content, Background and Rationale

There exists a profound interplay between the workings of the natural world and the laws and sensitivities of thought (cf. [1]). The most fundamental and firmly accepted parts of our general scientific knowledge of the world involve mathematical models (cf. [2]). And, as Chern begins his introduction to [3]: "While algebra and analysis provide the foundations of mathematics, geometry it at the core.". Geometry is the field of mathematics whose main source of intuition is human visual perception. So, it seems appropriate that geometry would contribute somewhat to a better understanding of visual perception.

In this note, a natural geometrical model is given for early human visual perception. Its rationale is that, essentially, looking at a luminance distribution means looking at the extrema of its fundamental shape–characteristics. Early human visual perception is hereby considered as both the result and the basis of the evolution and of the functioning of human perception as such. And human visual perception concerns the eye and brain–activities discussed by Gregory in [4]. This model originated, and below is also presented, in the context of visual illusions. Only a few related comments of a more general nature are made in this note. I do hope that some readers will enjoy to reflect upon this model in the full context of the fascinating science of vision.

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1. Introduction

It is a principle of visual perception that "we can see more than meets the eye". An inspiring general view on *the meaning of seeing* is offered in Gregory's book "Eye and Brain" [4].

In this note, I give *a model* (cf. [5]) for how "what meets the eye" is registered, or, in other words, *for early vision* as this has been developing through eye and brain–activities in biological species and individuals gifted with vision (§3). Its basis is a proper scientific description of what visual observations actually are, as was in particular so well formulated by Koenderinck and van Doorn [6] (§2). Paraphrasing Feynman's saying that "Nature speaks to us in the language of mathematics", this may illustrate that "*Nature likes to be looked at with geometers' eyes and brains*" in the sense that at least it explains visual illusions. In the light of this model, I made a few early reflections on the meaning of early vision, a.o. in relation with *artificial vision* and with *the visual arts* (§4). Taking this geometrical model seriously, following Gregory's approach [4, 7], I think of *visual preceptions* as *hypotheses which are seen in accordance with the information given by early vision* (§5).

2. On the Nature of Visual Observation

Consider a *static planar image* I that in the mathematical reality is given by a "luminance"–function F, (i.e. when x and y are Cartesian co-ordinates in the image–plane P, then $F(x,y)$ is the luminance at the points (x,y) in P), or, equivalently by a surface $N : z = F(x,y)$, the graph of the function F in the 3-dimensional Euclidean space E, (with Cartesian co-ordinates x, y and z). Such an image I is actually observed in the physical reality as a "*luminance"–function* L, or, equivalently, as a *surface* $M : Z = L(x,y)$ in E, which are respectively related to F and N by "smoothing" (say, by some kind of diffusion). And, whatever way one looks at vision, with every image I in P corresponds a "*visual–stimulus–surface*" M in E which describes what there is as a matter of fact to be seen in the image–plane P. In Figures 1 and 2 this is exemplified for (1): *a line–segment* S of *length* ℓ, and, for (2): an "*arrow*" A with S as shaft and thus of *length* $a = \ell$.

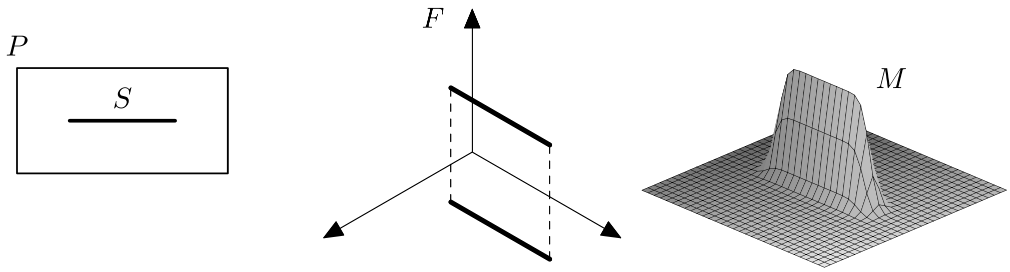


Figure 1.

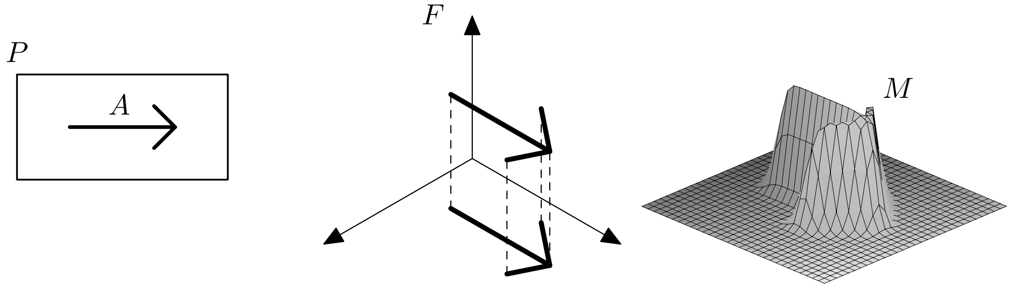


Figure 2.

As formulated by Koenderink and van Doorn cite4 in their comments on the nature of observation: the assumption on the existence ("at infinite resolution in scale space") of a "real" image I or F or N is nonsense, or still: *about the "real" image I in P we know nothing except for the observation L or M.* As far as I understand the literature on vision, many people busy with non-artistic studies on vision, right from the start of their speculations concerning perception, tend to confuse these physical and mathematical realities of observations. And, in my opinion, it are precisely such confusions that lie at the origin of a number of "mysterious phenomena" in the world of vision and in particular of many so-called visual illusions, as I will try to clarify in the sequal.

3. A Geometrical Model for Early Vision

In any case, when looking at a "real" image I in a plane P, all that we physically actually observe is what formally is well described by a visual–stimulus–

surface M in Euclidean 3-space E (at least in a *qualitative* way; later comments will indicate how to obtain a proper *quantitative* version of this in order to deal with the situation in a more subtle manner, a.o. in a non-Euclidean setting). A fundamental question in natural philosophy then is the following: "*What does our visual system register when observing a visual–stimulus–surface M in E?*".

This amounts to the question of how our visual system registers images I and should in a natural way be concerned with taking notice of important *characteristics of the shape of the corresponding surfaces M in E.* Differential geometry is the field of mathematics of which a main purpose ever since its origin has been to describe and understand the shape of such surfaces in Euclidean spaces and of their manifold generalisations. The most crucial *scalar-valued shape–characteristics* studied in differential geometry are various *notions of curvature*: from the principal curvatures of Euler–Meusnier, their product K (the Gauss curvature), their average H (the mean curvature of Germain), the average of their squares C (the Casorati curvature), etc., for surfaces M in E, over their affine, projective, isotropic,. . . , analogons, to curvatures like the ones of Riemann and B.Y. Chen for "surfaces" of arbitrary dimensions ([1, 3, 8]). And, in mathematics and in its applications, it is in general worthwhile to look at *certain extrema* of the scalar-valued quantities which are of relevance to the problems under investigation. In this general context, indeed, the most notable aspects of the shape of the visual–stimulus–surfaces M in E are given by certain extrema of its curvatures. And, some geometrical acquaintance with Euclidean surface–curvatures in combination with some reflection on the problem in view, yields the *Casorati curvature* C ([9, 10]) as a pretty suitable choice amongst the fundamental shape–characteristics of the visual–stimulus–surfaces M to consider further on. At this stage it should suffice to point out that the Casorati curvature of a surface M in E may well be the simplest kind of surface–curvature which, in agreement with rudimentary intuition, at each point measures the deviation of M from being planar, i.e. which accurately measures the extent to which M is curved as opposed to look like a flat plane.

At each point (x, y) in P the visual–stimulus–surface M in E corresponding to an image I has a Casorati curvature $C(x, y)$. So, every planar image I has a visual-stimulus-surface $M : z = L(x, y)$ in Euclidean 3-space E whose shape is significantly represented by its associated Casorati surface $z = C(x, y)$ in E. Figures 3 and 4 show the Casorati surfaces for the Examples 1 and 2, as well as the segments formed by their main relative extrema, which turn out to determine in P a *line–segment* $\tilde{S}$ and an *arrow* $\tilde{A}$ of respective *lengths* $\tilde{\ell}$ and $\tilde{a}$, whereby *essentially* $\tilde{a} < \tilde{\ell}$.

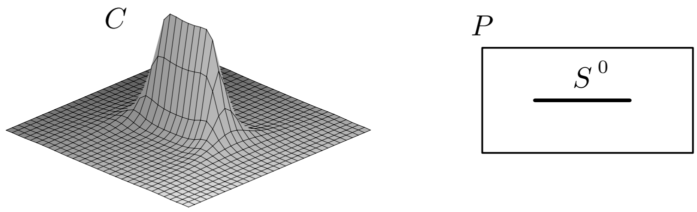


Figure 3.

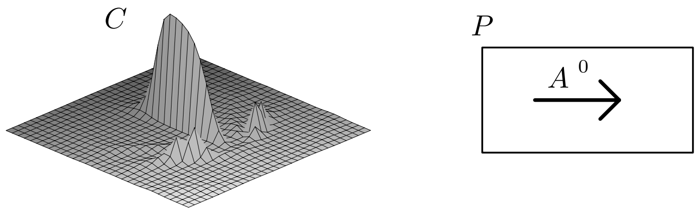
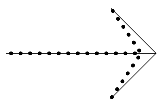


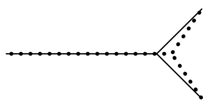
Figure 4.

The announced model for early vision then is the following. Our visual system "transforms" visual data, of which we think mathematically, by way of examples, for instance, as a line–segment S and an arrow A, and which we actually observe as visual–stimulus–surfaces M in E, to images consisting of extrema of the Casorati curvatures of M, which, in mathematical terms, in case of the examples, are a line–segment $\tilde{S}$ and an arrow $\tilde{A}$. In general and in short: *in early vision humans register images* $\tilde{I}$ *in* P *which are formed by points where the observations* M *made when looking at an image* I *in* P *have extremal curvatures.*

It seems to me that what is going on in many visual illusions, and in particular in the kind of visual illusions that can be assessed by comparing what is visually experienced with what can be materially measured (such as the illusions of Muller–Lyer, Hering, Oppell–Kundt, Źollner, Poggendorf, Judd, Ponzo, Titchener, Ebbinghaus, Ehrenstein, Orbison, Kanisza, Schumann, the Bristol café–wall, the corridor,... [4, 11, 12]), is the confrontation of the mathematical idealisation of real images I with the mathematical idealisation of their real early perceptions $\tilde{I}$. Or, put otherwise, it seems to me that such illusions arise by silently and dogmatically assuming that, when looking at an image I in a plane P, by early vision we should register precisely this image I. However when looking at an image I in P, the observation that our visual system makes is a corresponding visual–stimulus–surface M in E and, consequently, by early vision, we naturally register $\tilde{I}$, which is determined by the most characteristic features of M in E. For an arrow, by way of example, the comparison of $A(-)$ and $\tilde{A}(\ldots)$ thus qualitatively is as follows:



Also, one thus has, for instance $(I = -, \tilde{I} = \ldots)$:



and, as illustrated above for the Muller–Lyer one, similarly the other such illusions loose their mystery.

4. Some Comments and Remarks

Early vision is the basic eye and brain–activity which visually registers what there is actually to be seen. Qualitatively, for static planar images, this can be well modeled by properly appreciating the geometrical characteristics of the visual–stimulus–surfaces which are the natural formal description of visual observations. Of course, to turn the above model into an accurate *quantitative* model for early vision, a whole range of tests are needed to be carefully done, for instance, concerning *the calibration of the scaling and smoothing of* L *and* M, *the evaluation of* C in competition with other appropriate curvatures and *the modification of the Euclidean metric* in P and in the 3-space containing M in view of the anisotropy of the visual field and in view of the special character of the z-direction. Hereby, the available experimental expertise and data gathered over the years, in particular related to visual illusions, evidently are extremely valuable in themselves and for setting up additional useful experiments.

In all creatures gifted with vision, by inheritance and through personal experience, in both of which in a wide sense *cultural factors* guiding the meaning of vision may have great influence, the development of early vision leads to an individual visual registration system. When mutually compared, these individual systems may be more alike or more distinct according to the above factors, which can be reflected in the model by slight or more outspoken differences of the values of one or more geometrical quantities involved. For instance: the Muller–Lyer illusion turns out to be experienced less or more dramatically for individuals with significantly different cultural backgrounds, as described a.o. by Deregowski [13], which in the model could correspond to visual–stimulus–surfaces resulting from "steeper" or "slower" smoothings. To accordingly experimentally test a range of visual experiences with respect to consistency concerning similar viewers and with respect to other possible cross-cultural differences of early vision seems to be quite relevant. In particular, it could help to clarify *the eye and brain–processes that are involved in the formation and the evolution of the early vision–system.*

For reasons briefly mentioned before, the Casorati curvature of surfaces M in E is used at this stage rather than for instance the Gauss curvature K or the mean curvature H which rightfully deserved most attention in the traditional studies on surfaces in 3-dimensional Euclidean space till now; (K in the context of the intrinsic differential geometry and H in the context of the extrinsic differential geometry related to variational problems of physical surface–tension). Besides this, one might reflect a bit on the following comment concerning K, for which this can be done here more readily than for H. For *the visual illusion of Mach* (cf. [12]), the corresponding visual–stimulus–surface M in E is a cylinder whose rulers are horizontal straight lines, yielding that one of the principal curvatures is everywhere zero. Hence, the Gauss curvature K of M is also identically zero, although M is curved in the every day–meaning of this word according to the change in colour appearing in Mach's illusion. Therefore, K thus being a constant function in this case, or, equivalently, its graph being a horizontal plane in E, it is meaningless to consider its extrema. On the other hand, the extrema of for instance Casorati's curvature of this surface M do produce *the critical lines* in the illusion of Mach.

Concerning human vision and art, it can for instance be remarked that the eventual presence of weak curvature extrema of M sheds light on the phenomenon of *visual sensitivity* as experienced for instance in paintings, (cfr. [14, 15, 16]).

Concerning human vision and *artificial vision* it can for instance be remarked that given an image I it is quite possible to make machines correspondingly see the image $\tilde{I}$ that humans see when looking at I. And this could be given some further attention, for instance in neuroscience.

The above discussions are limited to the early vision of static 2D-images. For *static* $3D$*-images* and for *non-static* $2D$*- and* $3D$*-images* geometrical models can be made in a similar way, hereby involving curvatures of "generalized" surfaces.

5. An Analogy and an Opinion

Consider the following kind of analogy. When looking at photos of the 1919 expedition of Eddington, one can see on the dark disc of the sun a bright spot of the light emited by a star which is located behind the sun when viewed from our position on earth. This is no illusion or misperception: we see the light of this star on the disc of the sun only because it is there. It is a true observation of a physical reality which is well modeled by Einstein's theory of relativity. But when we interpret the same photo with the model of Newtonian physics it would be plausible to consider it to be an illusion. It is essentially through the adaptation of our intuition concerning situations in physics to the geometrical reality of a curved relativistic space–time that observations of phenomena such as for instance shown on Eddington's photos become really understandable. The consideration of visual illusions with the above model for early visual perception in the back of our heads can help to likewise adapt our intuition concerning visual situations.

In this respect, maybe the following could be of some help. Given an image I in a plane P, certainly in case I is not too complicated, one readily can mentally imagine or roughly draw corresponding visual–stimulus–surfaces M in E by thinking of “smoothing” or “filtering”. For those who are not so familiar with these processes, a useful impression of visual–stimulus–surfaces M corresponding to an image I can be obtained by thinking of waterfalls resulting from water running from levels of higher luminosity to levels of lower luminosity, whereby such levels can roughly be associated with I, (these levels are determinated by the function F, or still, these waterfalls originate by waterflows on the surface N in E). This gives us an impression of the surfaces M in E which describe the physical observation that we make when looking at an image I in a plane P. Then one should practice a bit in locating points and lines and curves formed in a consistent way by points where these surfaces are curved in an extremal manner compared with how they are curved in neighboring points. Thus one can get an idea of the image $\tilde{I}$ in P as given by early vision when looking at I. The above (let it be also rather rough) computer–figures shown in relation to the Muller–Lyer–illusion may be of some assistence in one’s first trials in this direction. Then one may enjoy looking in this way also at the other such visual illusions.

Above, I tried to explain my conviction of how in principle human early visual perceptions are related to physical reality and I claimed that the formation of the early vision-system is developed through eye and brain-activities. Now recall that *Gregory’s approach to the study of perception* is based on regarding perception “as constructing hypotheses... which may hit upon truth by producing symbolic structures matching physical reality” ([7]). In view of the above, this approach could somewhat liberally be reformulated as follows: *visual perception is the construction, inspired by images registered by early vision, of hypotheses which produce visual images that match physical reality in accordance with our previously acquired experiences of this reality.* It seems no idle spending of time to look in this way at what is going on in the universe of studies on visual perception as this for instance was recently surveyed in [17].

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