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δ-CONTINUOUS FUNCTIONS AND TOPOLOGIES ON FUNCTION SPACES

BY

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Abstract. In this paper we rename super continuous functions ([5]) as strongly δ-continuous functions and study the function space of δ-continuous & strongly δ-continuous functions via nets and filters. A necessary & sufficient condition in order that the δ-continuous convergence is the δ-convergence, is given.

Introduction

In this paper we use the concept of δ-continuous functions ([7]), N-closed subsets along with nearly compact spaces ([9]) and super continuous functions ([5]) what we call strongly δ-continuous functions.

Our endeavor is to introduce the concept of δ-continuous convergence and study its properties; study of a similar type for θ-continuous convergence was introduced by A. Di Concilio [3]; such study for strong δ-continuous convergence was done by B. K. Papadopoulos [6]. Secondly, we would use N-closed sets and regular open sets in studying the space of all δ-continuous functions $D(X,Y)$ and the space of all strongly δ-continuous functions $SD(X,Y)$ from X to Y by introducing a new topology.

Lastly, we introduce the δ-splitting and δ-conjoining topology and obtain their relation with types of convergence on $D(X,Y)$ (or $SD(X,Y)$).

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1. Prerequisites, Definitions and Theorems

Definition 1.1.([7]) Let X be a topological space. A set S in X is said to be regular open (respectively regular closed) if $Int.(cl.S) = S$ (respectively $Cl.(int.S) = S$). A point $x \in S$ is said to be a δ-cluster point of S if $S \cap U \neq \emptyset$, for every regular open set U containing x. The set of all δ-cluster point of S is called the δ-closure of S and is denoted by $[S]_\delta$. If $[S]_\delta = S$, then S is said to be δ-closed. The complement of a δ-closed set is called an δ-open set.

For every topological space (X, τ), the collection of all δ-open sets form a topology for X, which is weaker than τ. This topology τ^*, has a base consisting of all regular open sets in (X, τ).

Definition 1.2.([7]) A function $f : X \to Y$ is called a δ-continuous function iff for every regular open set V of Y, $f^{-1}(V)$ is δ-open in X.

The δ-continuity at a point $x \in X$ can also be defined as follows: A function $f : X \to Y$ is δ-continuous at a point $x \in X$ iff for every regular open nbd. V of $f(x)$ in Y, $\exists$ a δ-open nbd. U of x such that $f(U) \subseteq V$.

Definition 1.3.([5]) A function $f : X \to Y$ is strongly δ-continuous at a point $x \in X$ iff for any open nbd. V of $f(x)$ in Y, $\exists$ a δ-open nbd. U of x in X such that $f(U) \subseteq V$.

Definition 1.4.([2]) A set $A \subset (X, \tau)$ is said to be N-closed in X or simply N-closed, if for any cover of A by τ-open sets, there exists a finite sub-collection the interiors of the closure of which cover A; interiors and closures are of course w.r.t. τ.

A space (X, τ) is said to be nearly compact iff X is N-closed in X.

Definition 1.5.([1]) A space X is said to be semi regular if every point of the space has a fundamental system of regular open nbds.

Theorem 1.6.([1]) *A semi regular space is nearly compact iff it is compact.*

Observation 1.7. In view of Definition 1.1, it is obvious that (X, τ) is nearly compact iff (X, τ^*) is compact.

Observation 1.8. It is thus interesting to note that when a space is semi regular, concepts of near compactness and compactness are identical. Thus our interest lies in those spaces which are not semi regular.

Definition 1.9.([2]) A space (X, τ) is called locally nearly compact (l.n.c. in short) if for each point $x \in X$ there exists an open nbd. U of x such that $cl_X(U)$ is N-closed in X.

Definition 1.10.([8]) A space (X, τ) is said to be almost regular if any regularly closed set A and any singleton set $\{x\}$, disjoint from A can be strongly separated.

Lemma 1.11.([2]) *The following conditions are equivalent in a T_2 space X:*

(1) *X is locally nearly compact.*

(2) *For each N-closed set C in X and for each regular open neighbourhood U of C there exist an open set V such that $\bar{V}$ is N-closed and $C \subset V \subset \bar{V} \subset U$.*

Definition 1.12.([7]) A net $\{x_\lambda : \lambda \in \Lambda\}$ in (X, τ) is said to δ-converge to a point $x \in X$ iff every regular open nbd. of x contains the net eventually; we write $x_\lambda \xrightarrow{\delta} x$.

Definition 1.13.([7]) A filter $\Im$ on (X, τ) is said to δ-converge to a point $x \in X$ iff every $U \in \tau^*$, belongs to $\Im$.

Theorem 1.14.([7]) *A function $f : X \to Y$ is δ-continuous iff*

(1) *$\{f(x_\lambda)\}_{\lambda \in \Lambda}$ δ-converge to $f(x)$ for each $x \in X$ and each net $\{x_\lambda\}_{\lambda \in \Lambda}$ δ-converging to x.*

(2) *$f(\Im)$ δ-converges to $f(x)$ for each $x \in X$ and for each filter base $\Im$ δ-converging to x.*

Theorem 1.15.([4]) *Let (X, τ) and (Y, σ) be two topological space then $\tau^* \times \sigma^* = (\tau \times \sigma)^*$.*

Definition 1.16.([4]) The N-R topology on $D(X, Y)$ (or $SD(X, Y)$) is generated by the sets of the form $T(C, U) = \{f \in D(X, Y) : f(C) \subseteq U\}$, where C is a N-closed set in X and U a regular open set in Y.

Theorem 1.17. *If* (x_μ, y_ν) *is a net in* $X \times Y$ *where* X *&* Y *are topological spaces then* $(x_\mu, y_\nu) \xrightarrow{\delta} (x, y)$ *iff* $x_\mu \xrightarrow{\delta} x$ *&* $y_\nu \xrightarrow{\delta} y$.

The proof is obvious.

2. δ-Continuous Convergence in $D(X, Y)$

Definition 2.1. A net $\{f_\mu : \mu \in M\}$ in $D(X, Y)$ is said to be δ-continuously convergent to $f \in D(X, Y)$, if for any net $\{x_\lambda : \lambda \in \Lambda\}$ in X such that $x_\lambda \xrightarrow{\delta} x$, the net $f_\mu(x_\lambda) \xrightarrow{\delta} f(x)$.

Theorem 2.2. *A net* $\{f_\mu : \mu \in M\}$ *on* $D(X,Y)$ *is* δ*-continuously convergent to* $f \in D(X, Y)$ *iff for any* $x \in X$ *and any regular open nbd.* V *of* $f(x)$ *there is a* $\mu_0 \in M$ *and a* δ*-open nbd.* U *of* x *such that* $f_\mu(U) \subseteq V$ *for every* $\mu \in M \,\&\, \mu \geq \mu_0$.

Proof. Suppose that for a point $x \in X$ and a regular open nbd. V of $f(x)$ there exists a $\mu_0 \in M$ and a δ-open nbd. U of x such that $f_\mu(U) \subseteq V$ for all $\mu \in M \,\&\, \mu \geq \mu_0 \cdots (1)$.

Let $x_\lambda \xrightarrow{\delta} x$; then the net $\{x_\lambda\}_{\lambda \in \Lambda}$ is eventually in U i.e., $\exists\, \lambda_0 \in \Lambda$ such that for all $\lambda \geq \lambda_0$, $x_\lambda \in U \cdots (2)$.

Thus from (1) & (2) $f_\mu(x_\lambda) \in V$ for all $\lambda \geq \lambda_0$ & for all $\mu \geq \mu_0$ i.e., the net $f_\mu(x_\lambda)$ δ-converges to $f(x)$.

Conversely suppose that with $\{f_\mu : \mu \in M\}$ & $\{x_\lambda : \lambda \in \Lambda\}$ as in the statement of the theorem, $f_\mu(x_\lambda) \xrightarrow{\delta} f(x)$; we claim that for any regular open nbd. V of $f(x)$ there is a $\mu_0 \in M$ and a δ-open nbd. U of x such that $f_\mu(U) \subseteq V$ for every $\mu \in M \,\&\, \mu \geq \mu_0$; if not, for some regular open nbd. V of $f(x)$, for any $\mu \in M$ & any δ-open nbd. U of x, there is a $\mu' \in M$, $\mu' \geq \mu$ such that $f_{\mu'}(U) \not\subset V$ i.e., there exist a point $x_U \in U$ such that $f_{\mu'}(x_U) \notin V$.

Let $\mathcal{R}$ denote the class of all δ-open nbd. of x; we consider $(\mathcal{R}, \subset)$; obviously $(\mathcal{R}, \subset)$ is a directed set and $\{x_U : U \in \mathcal{R}\}$ is a net in X. Let $W \in \mathcal{R}$, then for any $U \subset W$ $(U \in \mathcal{R})$, $x_U \in U \subset W$ & thus $\{x_U : U \in \mathcal{R}\}$ δ-converges to x.

If possible, let $\{f_\mu(x_U)\}$ δ-converge to $f(x)$; then for some $\mu_0 \in M \,\&\, U_0 \in \mathcal{R}$, $f_\mu(x_U) \in V$ for all $\mu \geq \mu_0 \,\&\, U \subset U_0$; but for that μ_0, there exist a $\mu \geq \mu_0$ such that $f_\mu(U) \not\subset V$ i.e., $f_\mu(x_U) \notin V$, which is a contradiction. Hence our claim is true.

Note 2.3. In $D(X, Y)$ with some topology τ (say) we can thus have two types of convergence for a net $\{f_\mu : \mu \in M\}$. One is ordinary δ-convergence and the other is δ-continuous convergence.

Definition 2.4. With $P(X)$–the power set of a topological space X and $\mathcal{A} = \{A_\lambda : \lambda \in \Lambda\} \subset P(X)$ where Λ is a directed set, we define the δ-upper limit for $\mathcal{A}$ as the set of all points $x \in X$ such that for every $\lambda_0 \in \Lambda$ and every δ-open nbd. U of x in X there exist an element $\lambda \in \Lambda$ for which $\lambda \geq \lambda_0$ and $A_\lambda \cap U \neq \emptyset$. We denote the δ-upper limit for $\mathcal{A}$ by $\delta - \overline{\lim_{\Lambda}}(A_\lambda)$.

Theorem 2.5. *A net $\{f_\lambda : \lambda \in \Lambda\}$ in $D(X, Y)$ δ-continuously converges to $f \in D(X, Y)$ iff*

$$\delta - \overline{\lim_{\Lambda}}(f_\lambda^{-1}(K)) \subset f^{-1}(K) \cdots\cdots (\star),$$

for every δ-closed sub set K of Y.

Proof. Let $\{f_\lambda : \lambda \in \Lambda\}$ be a net in $D(X, Y)$ which δ-continuously converges to $f \in D(X, Y)$ & let K be an arbitrary δ-closed subset of Y. Let $x \in \delta - \overline{\lim_{\Lambda}}(f_\lambda^{-1}(K))$ and let W be an arbitrary regular open nbd. of $f(x)$ in Y. Since the net $\{f_\lambda : \lambda \in \Lambda\}$ δ-continuously converges to f, there exist a δ-open nbd. V of x in X and an element $\lambda_0 \in \Lambda$ such that $f_\lambda(V) \subseteq W$ for every $\lambda \in \Lambda$ with $\lambda \geq \lambda_0$ (by Theorem 2.2). On the other hand there exist an element $\lambda \geq \lambda_0$ such that $V \cap f_\lambda^{-1}(K) \neq \emptyset$. Hence $f_\lambda(V) \cap K \subseteq W \cap K \neq \emptyset$; i.e., $f(x)$ is a δ-cluster point of K; since K is δ-closed, $f(x) \in K$, $x \in f^{-1}(K)$.

Conversely let $\{f_\lambda : \lambda \in \Lambda\}$ be a net in $D(X, Y)$ and $f \in D(X, Y)$ such that the relation $(\star)$ holds for every δ-closed subset K of Y. We prove that the net $\{f_\lambda : \lambda \in \Lambda\}$ δ-continuously converges to f. Let $x \in X$ and let W be a regular open nbd. of $f(x)$ in Y. Let $K = Y \setminus W$; then K is δ-closed set. Now $x \notin f^{-1}(K)$ and by $(\star)$ $x \notin \delta - \overline{\lim_{\Lambda}}(f_\lambda^{-1}(K))$. Hence there exist an element $\lambda_0 \in \Lambda$ and a δ-open nbd. V of x in X such that $f_\lambda^{-1}(K) \cap V = \emptyset$ for every $\lambda \in \Lambda$ with $\lambda \geq \lambda_0$. Thus we have $V \subseteq X \setminus f_\lambda^{-1}(K) = f_\lambda^{-1}(Y \setminus K) \subset f_\lambda^{-1}(W)$. Thus $f_\lambda(V) \subseteq W$ for every $\lambda \in \Lambda$ with $\lambda \geq \lambda_0$. Hence the net $\{f_\lambda : \lambda \in \Lambda\}$ δ-continuously converges to f.

3. Joint δ-Continuity & Joint Strong δ-Continuity

By $N(X,Y)$, hence forth we shall denote the set of all $f \in Y^X$ such that the map f restricted over each N-closed set $C \subset X$ (i.e. $f|_C$) is δ-continuous; $SN(X,Y)$ shall denote the set of all $f \in Y^X$ such that $f|_C$, on each N-closed set $C \subset X$ is strongly δ-continuous.

Theorem 3.1.([2]) *Let X be a locally nearly compact T_2 space and Y be any topological space then $N(X,Y) = D(X,Y)$.*

Lemma 3.2.([9]) *If (X,σ) is a topological space such that $V \in \sigma$; then $\sigma^*|_V = (\sigma|_V)^*$.*

Lemma 3.3. *Let $f : X \to Y$ be a function such that $f|_{\bar{V}}$ is δ-continuous for an open set V with $\bar{V}$ N-closed. Then $f|_V$ is strongly δ-continuous.*

Proof. Let σ be the subspace topology for $\bar{V}$. Let W be an open set in Y. Then $f^{-1}(W) \cap \bar{V}$ is δ-open in $\bar{V}$ i.e., $f^{-1}(W) \cap \bar{V} \cap V = f^{-1}(W) \cap V$ belongs to $\sigma^*|_V$. By Lemma 3.2 $\sigma^*|_V = (\sigma|_V)^*$ i.e. $f^{-1}(W) \cap V$ is δ-open in V with its subspace topology which is inherited from the topology of $\bar{V}$ i.e. equivalently from the topology of X.

Lemma 3.4. *Let (X,τ) be a topological space and let $V \in \tau^*$; let $U \subset V$ be such that $U \in (\tau|_V)^*$; then $U \in \tau^*$.*

Its proof is obvious.

Theorem 3.5. *Let X be a locally nearly compact T_2 space and Y be any topological space then $SN(X,Y) = SD(X,Y)$.*

Proof. Since X is l.n.c. We can cover X by sets $\{V_x : x \in X\}$ such that $\bar{V}_x$ is N-closed in X; for any open set $U \subset Y$, $f^{-1}(U) = \cup\{f^{-1}(U) \cap \widehat{V}_x : x \in X\} \cdots\cdots (1)$, where $\widehat{V}_x = Int_X.cl_X V_x$. Since $f|_C$ is strongly δ-continuous for each N-closed set C, $f|_{\bar{V}}$ is strongly δ-continuous and so is $f|_{\widehat{V}}$ for every $\widehat{V} \in \{V_x : x \in X\}$. Thus $f^{-1}(U) \cap \widehat{V}_x$ is δ-open in $\widehat{V}_x$ for each x & so $f^{-1}(U) \cap \widehat{V}_x$ is δ-open in X (by Lemma 3.4) for each $x \in X$. From (1) $f^{-1}(U)$ is δ-open in X. Thus $SD(X,Y) = SN(X,Y)$.

Definition 3.6. Given a family F of functions on a topological space X to a topological space Y, it is reasonable to ask whether $\begin{array}{l} P: F \times X \to Y \\ (f,x) \ \to f(x) \end{array}$ is δ-continuous (strongly δ-continuous) for some topology τ on F (leading to a product topology on $F \times X$); if P is δ-continuous (strongly δ-continuous) for some topology τ on F, then τ is called a topology of joint δ-continuity (joint strong δ-continuity).

Definition 3.7. A topology for $F \subset Y^X$ is jointly δ-continuous (jointly strong δ-continuous) on N-closed sets iff it is jointly δ-continuous (jointly strong δ-continuous) on each N-closed sets in X i.e. iff $P|_{F\times K}$ is δ-continuous (strong δ-continuous) for each N-closed set $K \subset X$.

Theorem 3.8. *Let $F \subset Y^X$ be endowed with a topology which is jointly δ-continuous (jointly strong δ-continuous) on N-closed sets, then each member $f \in F$ is necessarily δ-continuous (strong δ-continuous) on each N-closed set K.*

Proof. Let U be a regular open (open) nbd. of $f(x)$ in Y; then there is a δ-open nbd. $T \times V$ of (f,x) in $F \times K$ such that $P(T \times V) \subset U$ (by Theorem 1.15 we can choose the δ-open nbds. for (f,x) as $T \times V$ where T is δ-open in F and V is δ-open K); now $(f,x) \in T \times V$ & so $P(f,x)$ for any $x \in V$ belongs to U i.e. $f(V) \subset U$ i.e. $f|_K$ is δ-continuous (strong δ-continuous).

Lemma 3.9. *If X&Y are topological spaces and A, B are N-closed sets in X&Y respectively and W is a δ-open set containing $A \times B$ in the product space $X \times Y$, then there are δ-open sets U&V respectively such that $A \subset U$, $B \subset V$, $U \times V \subset W$.*

Proof. It is straight forward in view of Theorem 1.15.

Lemma 3.10.([8]) *Every locally nearly compact T_2 space is almost regular.*

Lemma 3.11.([8]) *If X is a almost regular T_2 space and $A \subset X$ is N-closed with $A \subset U$ where U is a δ-open sets in X, then there exist a regular closed set V such that $A \subset V \subset U$.*

Theorem 3.12. *Each topology on $Z \subset Y^X$ which is jointly δ-continuous on each N-closed subset of X is larger than the N-R topology. If X is locally* *nearly compact T_2, Y is almost regular T_2 and each member of $Z \subset Y^X$ is δ-continuous on each N-closed subset of X, then the N-R topology (say τ) is jointly δ-continuous.*

Proof. The proof of the first part of the theorem is rather obvious.

Since X is l.n.c. T_2, every $f : X \to Y$ which is δ-continuous on each N-closed set of X is δ-continuous on X as well (by [4]). So what we have to show is that $\begin{array}{l} P : D(X,Y) \times X \to \ Y \\ \quad (f,x) \qquad \to f(x) \end{array}$ is δ-continuous. We take a regular open nbd. U of $f(x)$ in Y & take $(f,x) \in P^{-1}(U) \cdots (1)$. Since f is δ-continuous there exist a δ-open nbd. V of x in X such that $f(V) \subset U \cdots (2)$. Again X is l.n.c. T_2 so there exists a nbd. W of x in X such that $x \in W \subset \bar{W} \subset V \cdots (3)$, where $\bar{W}$ is N-closed in X. From (2) & (3) $f(\bar{W}) \subset U$; since δ-continuous image of an N-closed set is N-closed, $f(\bar{W})$ is N-closed in Y & contained in a regular open set U. Then by Lemma 3.11 there exist a regular closed set $\widehat{W}$ such that $f(\bar{W}) \subset \widehat{W} \subset U$; now $T(\bar{W}, \widehat{W})$ is a closed nbd. of f & for any $f \in T(\bar{W}, \widehat{W})$, $f(\bar{W}) \subset \widehat{W} \subset U$, and so is true for any $f \in Int.(T(\bar{W}, \widehat{W})) = \widehat{\widehat{W}}$, which is a δ-open nbd of f, then $(f,x) \in \widehat{\widehat{W}} \times \bar{W}$ & $P(\widehat{\widehat{W}} \times \bar{W}) \subset U$ giving P to be δ-continuous.

Definition 3.13. A topology τ on Y^X is δ-conjoining iff the map $\begin{array}{l} P : Y^X \times X \to \ Y \\ \quad (f,x) \ \to f(x) \end{array}$ is δ-continuous.

Observations 3.14. Obviously the N-R topology on $D(X,Y)$ is δ-conjoining (with the restriction imposed on X & Y).

Theorem 3.15. *Let $Z \subset D(X,Y)$; a topology τ on Z is δ-conjoining iff for any net $\{f_\mu\}_{\mu \in M}$ in Z which δ-converge to f in (Z,τ) implies that it δ-continuously converges to f in Z.*

Proof. Let τ be δ-conjoining i.e. $\begin{array}{l} P : Z \times X \to \ Y \\ \quad (f,x) \to f(x) \end{array}$ is δ-continuous and let $\{f_\mu\} \xrightarrow{\delta} f$: we have to show that the net $\{f_\mu\}_{\mu \in M}$ δ-continuously converges to f. We apply Theorem 2.2. Let U be a regular open nbd. of $f(x)$ in Y; since f is δ-continuous, there exists a δ-open nbd. V of x in X such that $f(V) \subset U$. Since P is δ-continuous, corresponding to that U there exist a δ-open nbd. $\widehat{W}$

of f and a δ-open nbd. V' of x such that $P(\widehat{W} \times V') \subset U$. If $W = V' \cap V$, then W is a δ-open nbd. of x (since intersection of two δ-open sets is δ-open). So $P(\widehat{W} \times W) \subset U \cdots (1)$, since the net $\{f_\mu\}_{\mu \in M}$ δ-converge to f, $\exists\ \mu_0 \in M$ such that $\mu \geq \mu_0$ implies $f_\mu \in \widehat{W}$; for all such f_μ by (1) $f_\mu(W) \subset U$. Thus by Theorem 2.2 $\{f_\mu\}_{\mu \in M}$ δ-continuously converges to f.

Conversely suppose that $\{f_\mu\} \xrightarrow{\delta} f$ in (Z, τ) implies that $\{f_\mu\}_{\mu \in M}$ δ-continuously converges to f; we have to show that the map $\begin{array}{ll} P:(Z,\tau) \times X \to & Y \\ (f,x) & \to f(x) \end{array}$ is δ-continuous. Let $\{(f_\mu, x_\nu) : \mu \in M, \nu \in \Lambda\}$ be a net in $(Z, \tau) \times X$ such that $(f_\mu, x_\nu) \xrightarrow{\delta} (f, x)$ in $Z \times X$. It will be sufficient to show that $P(f_\mu, x_\nu) \xrightarrow{\delta} P(f, x)$ i.e. $f_\mu(x_\nu) \xrightarrow{\delta} f(x)$ i.e. we have to show that $x_\nu \xrightarrow{\delta} x$ implies that $f_\mu(x_\nu) \xrightarrow{\delta} f(x)$, which is essentially the definition of δ-continuous convergence in Z. Thus the topology τ on Z is δ-conjoining.

Definition 3.16. A topology in Y^X is strongly δ-conjoining iff the map $\begin{array}{ll} P:Y^X \times X \to & Y \\ (f,x) & \to f(x) \end{array}$ is strongly δ-continuous.

Theorem 3.17. *Let $Z \subset SD(X, Y)$. A topology τ on Z is strongly δ-conjoining iff for any net $\{f_\mu\}_{\mu \in M}$ which converges* ([5]) *to f in (Z, τ) implies that it strongly δ-continuously converges to f in Z.*

The proof is similar to that of Theorem 3.15.

4. The Concept of Splitting Topology in Function Space

Given three spaces X, Y, Z, a function $\alpha(x, y) = z$ can be regarded as a map from $X \times Y$ to Z or as a family of maps $Y \to Z$ with X a parametric space. In this section it is an endeavor to check the effect of shifting from one point of view to other, on $D(X, Y)$ and $SD(X, Y)$.

For notation, let $\alpha : X \times Y \to Z$ be δ-continuous at $y \in Y$ for each fixed $x \in X$. The formula $[\widehat{\alpha}(x)](y) = \alpha(x, y) \cdots (1)$ defines for each fixed x, $\widehat{\alpha}(x) : Y \to Z$ which is δ-continuous i.e. $\widehat{\alpha}(x) \in D(Y, Z)$. So $\widehat{\alpha} : X \to D(Y, Z)$ is generated from the original mapping $\alpha : X \times Y \to Z$ as given.

Conversely given an $\widehat{\alpha} : X \to D(Y, Z)$, the formula (1) defines an $\alpha : X \times Y \to Z$ which is δ-continuous at $y \in Y$ for each fixed $x \in X$.

Definition 4.1. Two maps $\alpha : X \times Y \to Z$ & $\widehat{\alpha} : X \to D(Y, Z)$ related by the formula (1) are called associates.

The important feature of N-R topology on $D(Y, Z)$ gives :

Theorem 4.2. *Let $D(Y, Z)$ be endowed with N-R topology.*

(a) *If $\alpha : X \times Y \to Z$ is δ-continuous then $\widehat{\alpha} : X \to D(Y, Z)$ is strongly δ-continuous and hence δ-continuous.*

(b) *If $\widehat{\alpha} : X \to D(Y, Z)$ is δ-continuous and if Y is locally nearly compact, then $\alpha : X \times Y \to Z$ is also δ-continuous if Z is almost regular.*

Proof. (a) It is enough to show that for each $x_0 \in X$ and each subbasic set $T(A, V)$ of $D(Y, Z)$ in the N-R topology satisfying $\widehat{\alpha}(x_0) \in T(A, V)$ there is a δ-open nbd. U of x_0 in X such that $\widehat{\alpha}(U) \subset T(A, V)$. Equivalently, we must show that if $\alpha(x_0 \times A) \subset V$, then there are some δ-open nbd. U of x_0 with $\alpha(U \times A) \subset V$, where A is an N-set in Y. To this end we note that $x_0 \times A \subset \alpha^{-1}(V)$ and since α is δ-continuous, $\alpha^{-1}(V)$ is δ-open in $X \times Y$ & A is a N-closed set in Y so by Lemma 3.9 we get a δ-open nbd. U of x_0 with $(U \times A) \subset \alpha^{-1}(V)$. This completes the proof.

(b) We consider the sequence of mappings $X \times Y \xrightarrow{\widehat{\alpha}\times 1} D(Y, Z) \times Y \xrightarrow{P} Z$. Since $\widehat{\alpha}$ is δ-continuous so is $\widehat{\alpha} \times 1$ & P is δ-continuous by Theorem 3.12. Hence the combined map $P \circ (\widehat{\alpha} \times 1)$ is δ-continuous.
But $P(\widehat{\alpha}(x).(y)) = [\widehat{\alpha}(x)](y) = \alpha(x, y)$ and thus α is δ-continuous.

Definition 4.3. Let α and $\widehat{\alpha}$ as above be associates. A topology τ on $D(Y, Z)$ is called δ-splitting iff the δ-continuity of the map $\alpha : X \times Y \to Z$ implies the δ-continuity of the map $\widehat{\alpha} : X \to D(Y, Z)$ for every space X.

Theorem 4.4. *Let τ be a δ-splitting topology on $D(Y, Z)$ and $\{f_x\}_{x \in X} \subset D(Y, Z)$ δ-continuously converges to f; then $\{f_x\}$ δ-converges to f in τ ($D(Y, Z)$ is endowed with N-R topology.*

Proof. X is obviously a directed set and let us add a point ∞ to X such that $\infty \notin X$; to ascertain the natural order relation between ∞ and members of X, let us take $\infty \geq x$ for every $x \in X$. Let $\widehat{X} = X \cup \{\infty\}$; let $A_{x_0} = \{x \in \widehat{X} : x \geq x_0\}$. Obviously $\infty \in A_{x_0}$ for every $x_0 \in X$; we take $\{A_{x_0} : x_0 \in X\}$ to be the class of nbds. for ∞ and take every point of X to be open in $\widehat{X}$; then $\widehat{X}$ is topologized in such a manner that a net $\{x_\lambda\}$ in $\widehat{X}$ δ-converges to ∞ iff there exists λ_0 such that $x_\lambda \in A_{x_0}$ for all $\lambda \geq \lambda_0$ i.e. $x_\lambda \geq \lambda_0$ for all $\lambda \geq \lambda_0$; hence each $\{A_x : x \in X\}$ is open and hence δ-open as well.

Let $h : \widehat{X} \times Y \to Z$ be defined by setting $\begin{array}{l} h(x,y) = f_x(y), x \neq \infty, \\ \quad\quad\;\; = f(y)\;, x = \infty. \end{array}$ We claim that h is δ-continuous at (∞, y) for all $y \in Y$; let (x_λ, y_λ) be a net in $X \times Y$ δ-converging to (∞, y) in $X \times Y$. Let V be a δ-open nbd. of $f(y)$, since $\{y_\lambda\} \xrightarrow{\delta} y$ & $\{f_x\}$ δ-continuously converges to f, there exist λ_1 & x_0 such that for all $\lambda \geq \lambda_1 \,\&\, x \geq x_0$, $f_x(y_\lambda) \in V$; again $x_\lambda \to \infty$ implies that there exists λ_2 such that $\lambda \geq \lambda_2$ implies $x_\lambda \geq x_0$. Hence if $\lambda \geq \lambda_1, \lambda_2$ we have $\lambda \geq \lambda_0$ implies $f_{x_\lambda}(y_\lambda) \in V$ i.e. $h(x_\lambda, y_\lambda) = f_{x_\lambda}(y_\lambda) \xrightarrow{\delta} f(y)$. Hence h is δ-continuous at (∞, y); if $x \neq \infty$. Then $x_\lambda \xrightarrow{\delta} x$ iff $x_\lambda = x \;\forall\; \lambda \geq \lambda_0$ for some λ_0.

Then from δ-continuity of f_x at x it follows that $h(x, y_\mu) = f_x(y_\mu) \xrightarrow{\delta} f_x(y) = h(x,y)$ for any $y_\mu \xrightarrow{\delta} y$ in Y. Thus h is δ-continuous at (x,y) $\forall\; x \in X \,\&\, y \in Y$.

Now the δ-splitting property of h implies that $\widehat{h} : \widehat{X} \to D(Y,Z)$ is δ-continuous. Since every nbd. of ∞ contains the net $\{x\}$ eventually, $\{x\} \xrightarrow{\delta} x$ in $\widehat{X}$; again $\widehat{h}(x)[y] = h(x,y) = f_x(y) \;\forall\; x \in X \,\&\, y \in Y$ & hence $\widehat{h}(x) = f_x$ for every $x \in X$ & $\{f_x\} \xrightarrow{\delta} \widehat{h}(\infty)$ (as $\widehat{h}(x) \xrightarrow{\delta} \widehat{h}(\infty)$).

Now $[\widehat{h}(\infty)](y) = h(\infty, y) = f(y) \;\forall\; y \in Y$ & so $\widehat{h}(\infty) = f$. Thus $\{f_x\} \xrightarrow{\delta} f$.

Theorem 4.5. *If for any net $\{f_\mu\}$ of $(D(Y,Z), \tau)$, $\{f_\mu\}$ δ-continuously converges to f implies that $\{f_\mu\} \xrightarrow{\delta} f$ in $(D(Y,Z), \tau)$. Then τ is δ-splitting.*

Proof. Let $h : X \times Y \to Z$ be δ-continuous. Then for a net $\{x_\lambda\} \xrightarrow{\delta} x$ & $\{y_\mu\} \xrightarrow{\delta} y$, we have $(\widehat{h}(x_\lambda))(y_\mu) = h(x_\lambda, y_\mu) \xrightarrow{\delta} h(x,y) = (\widehat{h}(x))(y)$. Thus $\widehat{h}(x_\lambda) \xrightarrow{\delta} \widehat{h}(x)$ and it follows that $\widehat{h}$ is δ-continuous.

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