GENERALIZED DIFFERENCE PARANORMED STATISTICALLY CONVERGENT SEQUENCES DEFINED BY ORLICZ FUNCTION IN A LOCALLY CONVEX SPACE

BY

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Abstract. In this article we introduce the generalized difference paranormed sequence spaces \(c(\Delta^n, M, p, q)\), \(c_0(\Delta^n, M, p, q)\), \(m(\Delta^n, M, p, q)\) and \(m_0(\Delta^n, M, p, q)\) defined over a seminormed sequence space \((X, q)\). We study their different properties like completeness, solidity, symmetricity etc. We obtain some relations between these spaces as well as prove some inclusion results.

1. Introduction

Throughout the article \(w(X)\), \(c(X)\), \(c_0(X)\), \(\overline{c}(X)\), \(\overline{c}_0(X)\), \(\ell_\infty(X)\), \(m(X)\), \(m_0(X)\) will represent the spaces of all, convergent, null, statistically convergent, statistically null, bounded, bounded statistically convergent and bounded statistically null \(X\) valued sequence spaces, where \((X, q)\) is a seminormed space, seminormed by \(q\). For \(X = C\), the space of complex numbers, these represent the corresponding scalar valued sequence spaces. The zero sequence is denoted by \(\overline{0} = (\theta, \theta, \theta, \ldots)\), where \(\theta\) is the zero element of \(X\).

The notion of statistical convergence of sequences was introduced by Fast [5], Buck [1] and Schoenberg [25] independently. It is also found in Zygmund [33]. Later on it was studied from sequence space point of view and linked with summability theory by Fridy [6], Connor [2], Šalát [24], Maddox [16], Kolk [10],
The notion depends on the density of subsets of the set $N$ of natural numbers. A subset $E$ of $N$ is said to have density $\delta(E)$ if

$$\delta(E) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k)$$

exists, where $\chi_E$ is the characteristic function of $E$.

A sequence $(x_k)$ is said to be statistically convergent to $L$ (i.e., $(x_k) \in c$) if for every $\epsilon > 0$, $\delta(\{k \in N : |x_k - L| \geq \epsilon\}) = 0$. We write $x_k \xrightarrow{\text{stat}} L$ or $\text{stat-lim} x_k = L$.

Let $X$ be a linear space and $g : X \to R$ be such that

(i) $g(x) \geq 0$;
(ii) $x = \theta$ implies $g(x) = 0$;
(iii) $g(-x) = g(x)$;
(iv) $g(x + y) \leq g(x) + g(y)$;
(v) $g(\lambda_n x_n - \lambda x) \to 0$, as $n \to \infty$, whenever $\lambda_n \to \lambda$ and $x_n \to x$, for scalars $\lambda_n, \lambda$ and vectors $x, x_n$ (for all $n \in N$) $\in X$, then $g$ is called a paranorm and $(X, g)$ a paranormed space.

The notion of paranormed sequence space was introduced by Nakano [18] and Simons [26]. Later on it was further investigated by Maddox [14], Lascarides [12], Rath and Tripathy [21], Tripathy and Sen [32] and many others.

The notion of difference sequence space was introduced by Kizmaz [9]. It was generalized by Et and Colak [3] as follows:

Let $n$ be a non-negative integer, then

$$Z(\Delta^n) = \{ (x_k) \in w : (\Delta^n x_k) \in Z \},$$

for $Z = c, c_0$ and $\ell_{\infty}$, where $\Delta^n x_k = \Delta^{n-1} x_k - \Delta^{n-1} x_{k+1}, \Delta^0 x_k = x_k$ for all $k \in N$. Further the $n$th difference has the following binomial expression

$$\Delta^n x_k = \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} x_{k+\nu}. \quad (1)$$

Later on the generalized difference sequence space was investigated by Et and Nuray [4], Tripathy, Altin and Et [30], Tripathy, Et and Altin [31] and others.
An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$.

An Orlicz function $M$ is said to satisfy $\Delta_2$-condition for all values of $x$, if there exists a constant $K > 0$, such that $M(2x) \leq KM(x)$ for all $x \geq 0$.

Lindenstrauss and Tzafriri [13] used the idea of Orlicz function to construct the sequence space $\ell_M = \{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \}$, for some $\rho > 0$.

The space $\ell_M$ with the norm
$$
\|x\| = \inf\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\},
$$
becomes a Banach space, called as an Orlicz sequence space. The space $\ell_M$ is closely related to the space $\ell_p$ which is an Orlicz sequence space with $M(x) = |x|^p$ for $1 \leq p \leq \infty$.

If the convexity of the Orlicz function is replaced by $M(x+y) \leq M(x)+M(y)$, then this function is called as modulus function, introduced by Nakano [19] and studied by Ruckle [23], Maddox [15] and others.

**Remark 1.** An Orlicz function satisfies the inequality $M(\lambda x) \leq \lambda M(x)$ for all $\lambda$ with $0 < \lambda < 1$.

The following inequality will be used throughout the article. Let $p = (p_k)$ be a positive sequence of real numbers with $0 < p_k \leq \sup p_k = G, D = \max(1, 2^{G-1})$. Then for all $a_k, b_k \in C$ for all $k \in N$, we have
$$
|a_k + b_k|^{p_k} \leq D\{|a_k|^{p_k} + |b_k|^{p_k}\}.
$$

2. Definitions and Background

A sequence space $E$ is said to be solid (or normal) if $(\alpha_k x_k) \in E$, whenever $(x_k) \in E$ and for all sequences $(\alpha_k)$ of scalars with $|\alpha_k| \leq 1$ for all $k \in N$. 
A sequence space $E$ is said to be symmetric if $(x_k) \in E$ implies $(x_{\pi(k)}) \in E$, where $\pi(k)$ is a permutation of $N$.

A sequence space $E$ is said to be monotone if it contains the canonical pre-images of its step spaces i.e. $\chi.E \subset E$, where $\chi$ is the set of all sequences of zeros and ones.

We introduce the following definitions in this article. Let $p_k = (p_k)$ be a sequence of strictly positive real numbers. Then

$$
\sigma(\Delta^n, M, p, q) = \{(x_k) \in w(X) : [M(q(\frac{\Delta^n x_k - L}{\rho}))]^{p_k} \overset{\text{stat}}{\to} 0, \\
\text{for some } \rho > 0 \text{ and } L \in X \}
$$

$$
\sigma_0(\Delta^n, M, p, q) = \{(x_k) \in w(X) : [M(q(\frac{\Delta^n x_k}{\rho}))]^{p_k} \overset{\text{stat}}{\to} 0, \\
\text{for some } \rho > 0 \}.
$$

We procure the following definition for the sake of completeness.

$$
\ell_\infty(\Delta^n, M, p, q) = \{(x_k) \in w(X) : \sup_{k \geq 1} [M(q(\frac{\Delta^n x_k}{\rho}))]^{p_k} < \infty, \text{ for some } \rho > 0 \}.
$$

The following definition is introduced by Tripathy, Et and Altin [31].

$$
W(\Delta^n, M, p, q) = \{(x_k) \in w(X) : \lim_{j \to \infty} \sum_{k=1}^{j} [M(q(\frac{\Delta^n x_k - L}{\rho}))]^{p_k} = 0, \\
\text{for some } \rho > 0 \}.
$$

We write

$$
m(\Delta^n, M, p, q) = \sigma(\Delta^n, M, p, q) \cap \ell_\infty(\Delta^n, M, p, q)
$$

and

$$
m_0(\Delta^n, M, p, q) = \sigma_0(\Delta^n, M, p, q) \cap \ell_\infty(\Delta^n, M, p, q).
$$

For $p_k = 1$ for all $k \in N$, we write these spaces as $\sigma(\Delta^n, M, q)$, $\sigma_0(\Delta, M, q)$, $m(\Delta^n, M, q)$, $m_0(\Delta^n, M, q)$ and $\ell_\infty(\Delta^n, M, q)$. For $M(x) = x$, these reduce to the spaces $\sigma(\Delta^n, p, q)$, $\sigma_0(\Delta^n, p, q)$, $m(\Delta^n, p, q)$, $m_0(\Delta^n, p, q)$ and $\ell_\infty(\Delta^n, p, q)$, introduced and studied by Tripathy [29].

For $p_k = 1$ for all $k \in N$, $M(x) = x$ and $q(x) = |x|$, these spaces reduce to the spaces $\sigma(\Delta^n), \sigma_0(\Delta^n)$ introduced and studied by Et and Nuray [4]. For $n = 0,$
$M(x) = x$ and $q(x) = |x|$, these spaces reduce to the spaces $\tau(p)$, $\tau_0(p)$, $m(p)$ and $m_0(p)$ introduced and studied by Tripathy and Sen [32].

First we procure some results, those will help in establishing the results of this article.

**Lemma 1.** (Tripathy and Sen [32], Theorem 3) For two sequences $(p_k)$ and $(t_k)$ we have $m_0(p) \supseteq m_0(t)$ if and only if $\liminf_{k \in K} \frac{p_k}{t_k} > 0$, where $K \subseteq N$ such that $\delta(K) = 1$.

**Lemma 2.** (Tripathy and Sen [32], Theorem 4) Let $h = \inf p_k$ and $G = \sup p_k$, then the following are equivalent:

(i) $G < \infty$ and $h > 0$.

(ii) $m(p) = m$.

**Lemma 3.** (Tripathy and Sen [32], Lemma) Let $K = \{n_1, n_2, n_3, \ldots\}$ be an infinite subset of $N$ such that $\delta(K) = 0$. Let

$$T = \{(x_k) : x_k = 0 \text{ or } 1 \text{ for } k = n_i, i \in N \text{ and } x_k = 0, \text{ otherwise}\}.$$  

Then $T$ is uncountable.

The following result is well known, see for instance Kamtahm and Gupta [8], p.53.

**Lemma 4.** If a sequence space $E$ is solid, then $E$ is monotone.

### 3. Main Results

In this section we prove the results of this article involving $\tau(\Delta^n, M, p, q)$, $\tau_0(\Delta^n, M, p, q)$, $m(\Delta^n, M, p, q)$, $m_0(\Delta^n, M, p, q)$.

**Theorem 1.** $\tau(\Delta^n, M, p, q), \tau_0(\Delta^n, M, p, q), m(\Delta^n, M, p, q), m_0(\Delta^n, M, p, q)$ are linear spaces.

**Proof.** Let $(x_k), (y_k) \in \tau(\Delta^n, M, p, q)$. Then there exist $\rho_1, \rho_2$ positive real numbers and $L, J \in X$ such that

$$\left[ M(q \frac{\Delta^n x_k - L}{\rho_1}) \right]^{p_k} \xrightarrow{\text{stat}} 0$$

and

$$\left[ m(q \frac{\Delta^n y_k}{\rho_1}) \right]^{p_k} \xrightarrow{\text{stat}} 0$$

and

$$\left[ m_0(q \frac{\Delta^n y_k}{\rho_2}) \right]^{p_k} \xrightarrow{\text{stat}} 0$$

and

$$\left[ m_0(q \frac{\Delta^n y_k}{\rho_2}) \right]^{p_k} \xrightarrow{\text{stat}} 0$$
\[
\left[ M(q\left(\frac{\Delta^n y_k - J}{\rho_2}\right)) \right]^{p_k} \xrightarrow{\text{stat}} 0.
\]

Let \(\alpha, \beta\) be scalars and let \(\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)\). Then we have
\[
\left[ M\left(q\left(\frac{\Delta^n (\alpha x_k + \beta y_k) - (\alpha L + \beta J)}{\rho_3}\right)\right) \right]^{p_k} \\
\leq D\left\{ \left[ M(q\left(\frac{\Delta^n x_k - L}{\rho_1}\right)) \right]^{p_k} + \left[ M(q(\frac{\Delta^n y_k - J}{\rho_2})) \right]^{p_k} \right\} \xrightarrow{\text{stat}} 0, \text{ as } k \to \infty.
\]

Hence \(\mathfrak{T}(\Delta^n, M, p, q)\) is a linear space.

The rest of the cases will follow similarly.

**Theorem 2.** The spaces \(m(\Delta^n, M, p, q)\) and \(m_0(\Delta^n, M, p, q)\) are paranormed spaces, paranormed by
\[
g_{\Delta}(x) = \sum_{k=1}^{n} q(x_k) + \inf\{ \rho : \sup_k M(q(\frac{\Delta^n x_k}{\rho})) \leq 1, \rho > 0\},
\]
where \(M = \max(1, \sup p_k)\).

**Proof.** Clearly \(g_{\Delta}(x) = g_{\Delta}(-x); x = \theta\) implies \(\Delta^n x_k = \theta\) and as such \(M(q(\theta)) = 0\). Therefore \(g_{\Delta}(0) = 0\).

Let \((x_k)\) and \((y_k)\) be in any of the spaces in the statement. Then we have \(\rho_1, \rho_2 > 0\) such that
\[
\sup_k M(q(\frac{\Delta^n x_k}{\rho_1})) \leq 1
\]
and
\[
\sup_k M(q(\frac{\Delta^n y_k}{\rho_2})) \leq 1.
\]

Let \(\rho = \rho_1 + \rho_2\). Then by the convexity of \(M\), we have
\[
\sup_k M\left(q\left(\frac{\Delta^n (x_k + y_k)}{\rho}\right)\right) \\
\leq \left(\frac{\rho_1}{\rho_1 + \rho_2}\right) \sup_k M\left(q\left(\frac{\Delta^n x_k}{\rho_1}\right)\right) + \left(\frac{\rho_2}{\rho_1 + \rho_2}\right) \sup_k M\left(q\left(\frac{\Delta^n y_k}{\rho_2}\right)\right) \\
\leq 1.
\]
Hence we have,

\[
g_\Delta(x + y) = \sum_{k=1}^{n} q(x_k + y_k) + \inf \left\{ \rho \frac{p_k}{r} : \sup_k M \left(q \left(\frac{\Delta^n(x_k + y_k)}{\rho}\right)\right) \leq 1, \rho > 0 \right\}
\]

\[
\leq \sum_{k=1}^{n} q(x_k) + \inf \left\{ \rho_1 \frac{p_k}{r} : \sup_k M \left(q \left(\frac{\Delta^n x_k}{\rho_1}\right)\right) \leq 1, \rho_1 > 0 \right\}
\]

\[
+ \sum_{k=1}^{n} q(y_k) + \inf \left\{ \rho_2 \frac{p_k}{r} : \sup_k M \left(q \left(\frac{\Delta^n y_k}{\rho_2}\right)\right) \leq 1, \rho_2 > 0 \right\}
\]

\[
\leq g_\Delta(x) + g_\Delta(y).
\]

The continuity of the scalar multiplication follows from the following equality:

\[
g_\Delta(\lambda x) = \sum_{k=1}^{n} q(\lambda x_k) + \inf \left\{ \rho \frac{p_k}{r} : \sup_k M \left(q \left(\frac{\Delta^n(\lambda x_k)}{\rho}\right)\right) \leq 1, \rho > 0 \right\}
\]

\[
= |\lambda| \sum_{k=1}^{n} q(x_k) + \inf \left\{ \rho \frac{p_k}{r} : \sup_k M \left(q \left(\frac{\Delta^n x_k}{\rho}\right)\right) \leq 1, \rho > 0 \right\}, \text{ where } r = \frac{\rho}{|\lambda|}.
\]

Hence the spaces \(m(\Delta^n, M, p, q)\) and \(m_0(\Delta^n, M, p, q)\) are paranormed by \(g_\Delta\).

**Note.** The way the triangle inequality of norm or paranorm is established by most of the workers in their papers published so far from the linearity property of the space \(i.e.\) by taking \(\alpha = \beta = 1\) is not correct.

**Theorem 3.** Let \(M_1\) and \(M_2\) be two Orlicz functions satisfying \(\Delta_2\)-condition. Then

(i) \(Z(\Delta^n, M_1, p, q) \subseteq Z(\Delta^n, M_2 \circ M_1, p, q)\)

(ii) \(Z(\Delta^n, M_1, p, q) \cap Z(\Delta^n, M_2, p, q) \subseteq Z(\Delta^n, M_1 + M_2, p, q)\)

where \(Z = c, m, c_0\) and \(m_0\).

**Proof.** (i) We prove this part for \(Z = c_0\) and the rest of the cases will follow similarly. Let \((x_k) \in c_0(\Delta^n, M_1, p, q)\). Then for a given \(0 < \epsilon < 1\), we have there exists \(\rho > 0\) such that there exists a subset \(K\) of \(N\) with \(\delta(K) = 1\), where

\[
K = \left\{ k \in N : \left[ M_1 \left(q \left(\frac{\Delta^n x_k}{\rho}\right)\right) \right]^{p_k} < \frac{\epsilon}{B} \right\}
\]

and

\[
B = \max(1, \sup[M_2(1)]^{p_k}).
\]
Let $a_k = M_1(q(\frac{\Delta^nx_k}{\rho}))$, then $(a_k)^{p_k} < \frac{\epsilon}{B} < 1$ implies that $a_k < 1$. Hence we have by Remark 1,

$$(M_2 \circ M_1)(q(\frac{\Delta^nx_k}{\rho})) = M_2(a_k) \leq a_k M_2(1).$$

Thus

$$[M_2(a_k)]^{p_k} \leq B(a_k)^{p_k} < \epsilon.$$ 

Hence by (2) it follows that for a given $\epsilon > 0$, there exists $\rho > 0$ such that

$$\delta(\{k \in N : [M_2(M_1(q(\frac{\Delta^nx_k}{\rho})))]^{p_k} < \epsilon\}) = 1.$$ 

Therefore $(x_k) \in \overline{c_0}(\Delta^n, M_2 \circ M_1, p, q)$.

(ii) We prove this part for the case $Z = \overline{c_0}$ and the other cases will follow similarly. Let $(x_k) \in \overline{c_0}(\Delta^n, M_1, p, q) \cap \overline{c_0}(\Delta^n, M_2, p, q)$. Then using (2) it can be shown that $(x_k) \in \overline{c_0}(\Delta^n, M_1 + M_2, p, q)$. Hence

$$\overline{c_0}(\Delta^n, M_1, p, q) \cap \overline{c_0}(\Delta^n, M_2, p, q) \subseteq \overline{c_0}(\Delta^n, M_1 + M_2, p, q).$$

This completes the proof of the theorem.

**Theorem 4.** Let $n \geq 1$, then for all $0 \leq i \leq n$, $Z(\Delta^i, M, p, q) \subseteq Z(\Delta^n, M, p, q)$, where $Z = \overline{c}, m, \overline{c_0}$ and $m_0$.

**Proof.** We establish it for $\overline{c_0}(\Delta^n, M, p, q) \subseteq \overline{c_0}(\Delta^n, M, p, q)$. It follows from the following inequality

$$[M(q(\frac{\Delta^nx_k}{\rho}))]^{p_k} \leq D\{[M(q(\frac{\Delta^{n-1}x_k}{\rho}))]^{p_k} + [M(q(\frac{\Delta^{n-1}x_k}{\rho}))]^{p_k}\}$$

that $(x_k) \in \overline{c_0}(\Delta^{n-1}, M, p, q)$ implies $(x_k) \in \overline{c_0}(\Delta^n, M, p, q)$.

On applying the principle of induction it follows that

$$\overline{c_0}(\Delta^i, M, p, q) \subseteq \overline{c_0}(\Delta^n, M, p, q), \text{ for } i = 0, 1, 2, \ldots, n - 1.$$ 

The proof for the rest of the cases will follow similarly.

To show that the inclusions are strict, consider the following example.

**Example 1.** let $q(x) = |x|$, $M(x) = x$ and $p_k = 1$ for all $k \in N$. Then the sequence $(x_k) = (k^{n-1}) \in Z(\Delta^n)$ but $(x_k) \notin Z(\Delta^{n-1})$ for $Z = \overline{c_0}$ and $m_0$. 
since $\Delta^nx_k = 0$ and $\Delta^{n-1}x_k = (-1)^{n-1}(n-1)!$ for all $k \in N$. Under the above restrictions on $q, M$ and $(p_k)$ and for $n = 1$, consider the sequence $(x_k) = (k)$. Then $(x_k) \in Z(\Delta)$, but $(x_k) \notin Z$ for $Z = c$ and $m$.

**Theorem 5.** For any two sequences $p = (p_k)$ and $t = (t_k)$ of positive real numbers and for any two seminorms $q_1$ and $q_2$ on $X$ we have

$$Z(\Delta^n, M, p, q_1) \cap Z(\Delta^n, M, p, q_2) \neq \Phi,$$

where $Z = c, m, c_0$ and $m_0$.

**Proof.** The proof follows from the fact that the zero element belongs to each of the classes of sequence spaces involved in the intersection.

The proof of the following result is a routine work.

**Proposition 6.** Let $M$ be an Orlicz function, $q_1$ and $q_2$ be two seminorms on $X$. Then

(i) $Z(\Delta^n, M, p, q_1) \cap Z(\Delta^n, M, p, q_2) \subseteq Z(\Delta^n, M, p, q_1 + q_2)$, where $Z = c, m, c_0$ and $m_0$.

(ii) $\overline{c}(\Delta^n, M, p, q_1) \subseteq \overline{c}(\Delta^n, M, p, q_1)$.

(iii) $m(\Delta^n, M, p, q_1) \subseteq m(\Delta^n, M, p, q_1)$.

(iv) If $q_1$ is stronger than $q_2$, then

$$Z(\Delta^n, M, p, q_1) \subseteq Z(\Delta^n, M, p, q_2),$$

where $Z = c, m, c_0$ and $m_0$.

**Theorem 7.** The spaces $Z(\Delta^n, M, p, q)$, are not solid for $n > 0$, where $Z = c, m, c_0$ and $m_0$.

**Proof.** To show that the spaces are not solid in general, consider the following examples.

**Example 2.** Let $n = 2$, $X = \ell_\infty$, $q(x) = \sup|x^i|$, where $(x) = (x^i) \in \ell_\infty$. Let $M(x) = x$ and $p_k = 1$ for all $k$ odd and $p_k = 2$ for all $k$ even. Consider the sequence $(x_k)$, where $x_k = (x^i_k) \in \ell_\infty$ be defined by $(x^i_k) = (k, k, k, \ldots)$ for each fixed $k \in N$. Then $\Delta^2x_k = (0, 0, 0, \ldots)$ for all $k \in N$. Hence $(x_k) \in Z(\Delta^2, p, q)$.
for $Z = r$ and $m$. Let $\alpha_k = (-1)^k$, then $(\alpha_k x_k) \notin Z(\Delta^2, p, q)$ for $Z = r$ and $m$.
Thus $Z(\Delta^2, p, q)$ for $Z = r$ and $m$ is not solid.

**Example 3.** Let $n = 2$, $X = c$, $q(x) = \sup_i |x_i|$, where $x = (x_i) \in c$. Let $M(x) = x^2$ and $p_k = 2$ for all $k$ odd and $p_k = 3$ for all $k$ even. Consider the sequence $(x_k)$, where $x_k = (x_k^i) \in c$ be defined by $(x_k^i) = (1, 1, 1, \ldots)$ for each fixed $k \in N$. Then $\Delta^2 x_k = (0, 0, 0, \ldots)$ for all $k \in N$. Hence $(x_k) \in Z(\Delta^2, M, p, q)$ for $Z = r$ and $m_0$. Let $\alpha_k = (-1)^k$, then $(\alpha_k x_k) \notin Z(\Delta^2, M, p, q)$.
Therefore $Z(\Delta^2, M, p, q)$ is not solid for $Z = r$ and $m_0$.

**Remark 2.** For $n = 0$, the spaces $Z(M, p, q)$ will be solid for $Z = r$ and $m_0$ and are as such monotone.

**Theorem 8.** The spaces $Z(\Delta^n, M, p, q)$, are not symmetric for $n > 0$, where $Z = r, m, r$, and $m_0$.

**Proof.** To show that the spaces are not symmetric, consider the following examples.

**Example 4.** Let $n = 3$, $X = R^2$, $q(x) = \max \{|x_1|, |x_2|\}$, for $x = (x_1, x_2) \in R^2$. Let $M(x) = x_1$, $p_k = 1$ for all $k$ odd and $p_k = 2$ for all $k$ even. Consider the sequence $(x_k)$ defined by $(x_k) = (k, k)$ for each fixed $k \in N$. Then $(x_k) \in Z(\Delta^3, p, q)$ for $Z = r$ and $m$. Let $(y_k)$ be a rearrangement of $(x_k)$, which is defined as follows:

$$(y_k) = \{x_1, x_2, x_3, x_9, x_5, x_6, x_25, x_7, x_36, x_8, x_49, x_10, \ldots\}. $$

Then $(y_k) \notin Z(\Delta^3, p, q)$ for $Z = r$ and $m$.

**Example 5.** Let $n = 2$, $p_k = 1$ for all $k$ odd and $p_k = 2^{-1}$ for all $k$ even. Let $X = C^3$ and $q(x) = \max \{|x_1|, |x_2|, |x_3|\}$, where $(x) = (x_1, x_2, x_3) \in C^3$. Let $M(x) = x^4$. Consider the sequence $(x_k)$ defined by $(x_k) = (2, 2, 2)$ for $(2i - 1)^2 \leq k < (2i)^2$, $i \in N$ and $(x_k) = (5, -1, 7)$, otherwise. Then $(x_k) \in Z(\Delta^2, M, p, q)$ for $Z = r$ and $m_0$. Consider $(y_k)$ the rearrangement of $(x_k)$ defined by

$$(y_k) = \{x_1, x_2, x_4, x_9, x_5, x_6, x_25, x_7, x_36, x_8, x_49, x_10, \ldots\}. $$

Then $(y_k) \notin Z(\Delta^2, M, p, q)$ for $Z = r$ and $m_0$. Thus the spaces $Z(\Delta^n, M, p, q)$, where $Z = r, m, r$, and $m$, and $\ell_\infty$ are not symmetric in general.
Taking $y_k = M(q(\Delta^n x_k / \rho))$ for all $k \in N$, we have the following three results, those follow from Lemma 1 and Lemma 2.

**Proposition 9.** For two sequences $(p_k)$ and $(t_k)$ we have $m_0(\Delta^n, M, p, q) \supseteq m_0(\Delta^n, M, t, q)$ if and only if $\lim \inf_{k \in K} \frac{p_k}{t_k} > 0$, where $K \subseteq N$ such that $\delta(K) = 1$.

The following result is a consequence of the above result.

**Corollary 10.** For two sequences $(p_k)$ and $(t_k)$ we have $m_0(\Delta^n, M, p, q) = m_0(\Delta^n, M, t, q)$ if and only if $\lim \inf_{k \in K} p_k t_k > 0$ and $\lim \inf_{k \in K} \frac{t_k}{p_k} > 0$, where $K \subseteq N$ such that $\delta(K) = 1$.

**Proposition 11.** Let $h = \inf p_k$ and $G = \sup p_k$, then the following are equivalent:

(i) $G < \infty$ and $h > 0$.

(ii) $m(\Delta^n, M, p, q) = m(\Delta^n, M, q)$.

**Theorem 12.** Let $p = (p_k)$ be a sequence of non-negative bounded real numbers such that $\inf p_k > 0$. Then $m(\Delta^n, M, p, q) = W(\Delta^n, M, p, q) \cap \ell_\infty(\Delta^n, M, p, q)$.

**Proof.** Let $(x_k) \in W(\Delta^n, M, p, q) \cap \ell_\infty(\Delta^n, M, p, q)$. Then for a given $\epsilon > 0$, we have

$$\sum_{k=1}^{j} \left[ M(q\left(\frac{\Delta^n x_{k} - L}{\rho}\right)) \right]^{p_k} \geq \text{Card}\left\{ k \leq j : \left[ M(q\left(\frac{\Delta^n x_{k} - L}{\rho}\right)) \right]^{p_k} \geq \epsilon \right\} \cdot \epsilon.$$ 

From the above inequality it follows that $(x_k) \in m(\Delta^n, M, p, q)$.

Conversely let $(x_k) \in m(\Delta^n, M, p, q)$. Let $\rho > 0$ be such that

$$\left[ M(q\left(\frac{\Delta^n x_{k} - L}{\rho}\right)) \right]^{p_k} \text{ stat} \to 0.$$ 

For a given $\epsilon > 0$, let $B = \sup_{k} \left[ M(q\left(\frac{\Delta^n x_{k} - L}{\rho}\right)) \right]^{p_k} < \infty$.

Let $L_j = \{ k \leq j : \left[ M(q\left(\frac{\Delta^n x_{k} - L}{\rho}\right)) \right]^{p_k} \geq \frac{\epsilon}{2} \}$.

Since $(x_k) \in m(\Delta^n, M, p, q)$, so $\frac{\text{Card}(L_j)}{j} \to 0$, as $j \to \infty$. Let $n_0 > 0$ be such that $\frac{\text{Card}(L_j)}{j} < \frac{\epsilon}{2B^\rho}$ for all $j > n_0$. Then for all $j > n_0$, we have

$$\frac{1}{j} \sum_{k=1}^{j} \left[ M(q\left(\frac{\Delta^n x_{k} - L}{\rho}\right)) \right]^{p_k}$$

$$\leq 1 + \sum_{j=n_0}^{\infty} \frac{\text{Card}(L_j)}{j} \cdot \frac{\epsilon}{2B^\rho}$$

Hence, $m(\Delta^n, M, p, q) = W(\Delta^n, M, p, q) \cap \ell_\infty(\Delta^n, M, p, q)$.
\[= \frac{1}{j} \sum_{k \not\in L_j} [M(q(\frac{\Delta^nx_k - L}{\rho}))]^{\beta_k} + \frac{1}{j} \sum_{k \in L_j} [M(q(\frac{\Delta^nx_k - L}{\rho}))]^{\beta_k}\]
\[\leq \frac{j - \text{Card}\{L_j\}}{j} \frac{\epsilon}{2} + \frac{\text{Card}\{L_j\}}{j} \cdot B^H\]
\[\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\]

Hence \((x_k) \in W(\Delta^n, M, p, q) \cap \ell_\infty(\Delta^n, M, p, q)\).
This completes the proof of the theorem.

The following result is a consequence of the above theorem.

**Corollary 13.** Let \((p_k)\) and \((t_k)\) be two bounded sequences of real numbers such that \(\inf p_k > 0\) and \(\inf t_k > 0\). Then

\[W(\Delta^n, M, p, q) \cap \ell_\infty(\Delta^n, M, p, q) = W(\Delta^n, M, t, q) \cap \ell_\infty(\Delta^n, M, t, q).\]

**Theorem 14.** Let \((X, q)\) be a complete seminormed space, then the spaces \(m(\Delta^n, M, p, q)\) and \(m_0(\Delta^n, M, p, q)\) are complete.

**Proof.** We prove the result for the case \(m_0(\Delta^n, M, p, q)\) and for the other space it will follow on applying similar arguments. Let \((x^i)\) be a Cauchy sequence in \(m_0(\Delta^n, M, p, q)\). Let \(\delta > 0\) be fixed and \(r > 0\) be such that for a given \(0 < \epsilon < 1, \frac{\epsilon}{r\delta} > 0\), and \(r\delta \geq 1\). Then there exists a positive integer \(n_0\) such that \(g_\Delta(x^i - x^j) < \frac{\epsilon}{r\delta}\), for all \(i, j \geq n_0\).

\[\Rightarrow \sum_{k=1}^{n} q(x^i_k - x^j_k) + \inf \{\rho^{\beta_k} : \sup M(q(\frac{\Delta^nx_k - \Delta^nx^j_k}{\rho})) \leq 1, \rho > 0\} < \frac{\epsilon}{r\delta}, \quad (3)\]
for all \(i, j \geq n_0\).

From (3) we have \((x^i_k)_{i=1}^{\infty}\) is a Cauchy sequence in \(X\) for each \(k = 1, 2, 3, \ldots, n\).
Hence are convergent in \(X\), for each fixed \(k = 1, 2, 3, \ldots, n\). Let

\[\lim_{i \to \infty} x^i_k = x_k, \text{ for each } k = 1, 2, 3, \ldots, n.\]  \quad (4)

From (3) we have

\[|M(q(\frac{\Delta^nx_k - \Delta^nx^j_k}{\rho}))| \leq M(\frac{r\delta}{2}), \text{ for all } i, j \geq n_0 \text{ and all } k \in N,\]
\[\Rightarrow q(\Delta^nx_k - \Delta^nx^j_k) < \epsilon, \text{ for all } i, j \geq n_0 \text{ and all } k \in N.\]
Hence \((\Delta^n(x^i_k))_{i=1}^\infty\) for all \(k \in N\), is a Cauchy sequence in \(X\) and hence are convergent in \(X\). Let \(\lim_{i \to \infty} \Delta^n x^i_k = y_k\).

Now for \(k = 1\), by eq (1), (4) and \(\lim_{i \to \infty} \Delta^n x^i_1 = y_1\), we have \(\lim_{i \to \infty} x^i_{k+1} = x_{k+1}\) exists. Proceeding in this way inductively we have \(\lim_{i \to \infty} x^i_k = x_k\), for each \(k \in N\).

Now using the continuity of \(M\) and applying the standard techniques, we have for all \(i \geq n_0\),

\[
\lim_{j \to \infty} \sum_{k=1}^n q(x^i_k - x^j_k) + \inf \left\{ \rho \frac{n}{M} : \lim_{j \to \infty} \sup_k M(q(\Delta^n x^i_k - \Delta^n x^j_k)) \leq 1, \rho > 0 \right\} < \frac{2\epsilon}{r_0}
\]

\[
\Rightarrow \sum_{k=1}^n q(x^i_k - x_k) + \inf \left\{ \rho \frac{n}{M} : \sup_k M(q(\Delta^n x^i_k - \Delta^n x_k)) \leq 1, \rho > 0 \right\} < \frac{2\epsilon}{r_0}
\]

\[
\Rightarrow g_\Delta(x^i - x) < \frac{2\epsilon}{r_0}.
\]

Hence \((x^i - x) \in m_0(\Delta^n, M, p, q)\). Since \((x^i) \in m_0(\Delta^n, M, p, q)\) and \(m_0(\Delta^n, M, p, q)\) is a linear space, so we have \(x = x^i - (x^i - x) \in m_0(\Delta^n, M, p, q)\). Hence \(x \in m_0(\Delta^n, M, p, q)\) is a closed subspace of \(\ell_\infty(\Delta^n, M, p, q)\).

Since the inclusion relations \(m(\Delta^n, M, p, q) \subset \ell_\infty(\Delta^n, M, p, q)\) and \(m_0(\Delta^n, M, p, q) \subset \ell_\infty(\Delta^n, M, p, q)\) are strict, we have the following result.

**Corollary 15.** The spaces \(m(\Delta^n, M, p, q)\) and \(m_0(\Delta^n, M, p, q)\) are nowhere dense subsets of \(\ell_\infty(\Delta^n, M, p, q)\).

The following result is obvious in view of Lemma 3.

**Proposition 16.** The spaces \(m(\Delta^n, M, p, q)\) and \(m_0(\Delta^n, M, p, q)\) are not separable.

### 4. Relation with Distribution Theory

Let us consider the following example.

**Example 6.** At a party \(m\) number of men, numbered 1 to \(m\) threw their hats to the center of a room. The hats are mixed up and each man randomly
selects one. Find the expected number of men bearing the numbers in $A$, where $A = \{1, 2, 3, \ldots, m\} - \{i \leq m : i = k^2, k \in N\}$ select their own hats.

**Solution.** Let $X$ denote the number of men that select their own hats, bearing the numbers in the set $A$. Then

\[
X = X_2 + X_3 + X_5 + \cdots + X_m, \text{ if } \sqrt{m} \notin N
\]
\[
= X_2 + X_3 + X_5 + \cdots + X_{m-1}, \text{ if } \sqrt{m} \in N.
\] (5)

Now the $i^{th}$ man is equally likely to select any of the $m$ hats, it follows that

\[
P\{X_i = 1\} = P\{i^{th} \text{ man selects his own hat}\} = \frac{1}{m}
\]

and so,

\[
E[X_i] = 1 \cdot P\{X_i = 1\} + 0 \cdot P\{X_i = 0\} = \frac{1}{m}.
\]

From (5) we have

\[
E[X] = E[X_2] + E[X_3] + E[X_5] + \cdots + E[X_m], \text{ if } \sqrt{m} \notin N
\]
\[
= E[X_2] + E[X_3] + E[X_5] + \cdots + E[X_{m-1}], \text{ if } \sqrt{m} \in N.
\]

Since Card\{$i \leq m : i = k^2, k \in N$\} $\leq \sqrt{m}$, so we have

\[
\frac{m - \sqrt{m}}{m} \leq E[X] \leq 1.
\]

In the limiting case, as $m \to \infty$, we have $E[X] = 1$.

Hence the expected number of man not bearing number in the set $A$, who selects his own hat is 1 in the limiting case.

**Remark 3.** In the above example, for a given $\epsilon > 0$, if one considers

\[
\left[ M(g(\frac{\Delta^n x_k - L}{\rho})) \right]^{p_k} < \epsilon
\] (6)

instead of the man selecting his own hat, then one will get the expected number of terms of the sequence $\langle x_k \rangle$ satisfying the condition (6).

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References


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