ON GENERALIZATIONS OF BIHARI’S INEQUALITY

BY

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Abstract. In this paper, we establish some generalizations of the Bihari’s inequality which will be more useful in certain applications. Two independent variable generalizations of the main results, the discrete analogues and applications of one of our results are also given.

1. Introduction

In 1971 I. Göri [3] (see, also [2, p.10]) proved the following generalization of the Bihari’s inequality.

Theorem G. Suppose that \( u(t) \) and \( \beta(t) \) are continuous and nonnegative on \([t_0, \infty)\). Let \( f(t), g(u) \) and \( \alpha(t) \) be differentiable functions with \( f \) nonnegative, \( g \) positive and nondecreasing and \( \alpha \) nonnegative and nonincreasing. Suppose that

\[
  u(t) \leq f(t) + \alpha(t) \int_{t_0}^{t} \beta(s)g(u(s))ds.  \tag{1.1}
\]

If

\[
  f'(t) \left\{ \frac{1}{g(\eta(t))} - 1 \right\} \leq 0 \quad \text{on } [t_0, \infty), \tag{1.2}
\]

for every nonnegative continuous function \( \eta \), then

\[
  u(t) \leq G^{-1} \left[ G(f(t_0)) + \int_{t_0}^{t} \left[ \alpha(s)\beta(s) + f'(s) \right] ds \right],  \tag{1.3}
\]

where

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\[ G(\delta) = \int_{\epsilon}^{\delta} \frac{ds}{g(s)}, \quad \epsilon, \delta > 0, \]

and (1.3) holds for all values of \( t \) for which the function

\[ \delta(t) = G(f(t_0)) + \int_{t_0}^{t} [\alpha(s)\beta(s) + f'(s)]ds \]

belongs to the domain of the inverse function \( G^{-1} \).

Indeed, by choosing \( \eta(t) = u(t) \), from (1.2) it is easy to observe that

\[ f(t) \leq f(t_0) + \int_{t_0}^{t} f'(s)g(u(s))ds. \quad (1.4) \]

Using (1.4) in (1.1) we get

\[ u(t) \leq f(t_0) + \int_{t_0}^{t} [\alpha(t)\beta(s) + f'(s)]g(u(s))ds. \quad (1.5) \]

In fact (1.5) is a general version of Bihari’s inequality (see [1, 4]). However, it seems, the condition (1.2) on the known function \( f(t) \) restrict the class of functions for which inequality in Theorem G is applicable. The main purpose of the present paper is to establish explicit bounds on the general version of (1.5) which will be more convenient than Theorem G in applications. Two independent variable generalizations of the main results, the discrete analogues and applications of one of our results are also given.

2. Statement of Results

In what follows, \( R \) denotes the set of real numbers, \( R_{+} = [0, \infty), \ N_{0} = \{0, 1, 2, \ldots\} \) are the given subsets of \( R \) and ' denotes the derivative. The partial derivatives of a function \( v(x, y) \), \( x, y \in R \) with respect to \( x, y \) and \( xy \) are denoted by \( D_1v(x, y), D_2v(x, y) \) and \( D_1D_2v(x, y) = D_2D_1v(x, y) \) respectively. For functions \( w(m), z(m, n), m, n \in N_{0} \), we define the operators \( \Delta, \Delta_1, \Delta_2 \) by

\[ \Delta w(m) = w(m + 1) - w(m), \]
\[ \Delta_1 z(m, n) = z(m + 1, n) - z(m, n), \]
\[ \Delta_2 z(m, n) = z(m, n + 1) - z(m, n), \]
respectively and
\[ \Delta_2 \Delta_1 z(m, n) = \Delta_2(\Delta_1 z(m, n)). \]

We denote by
\[ G_1 = \{(t, s) \in \mathbb{R}^2_+ : 0 \leq s \leq t < \infty\}, \]
\[ G_2 = \{(x, y, s, t) \in \mathbb{R}^4_+ : 0 \leq s \leq x < \infty, 0 \leq t \leq y < \infty\}, \]
\[ H_1 = \{(m, n) \in \mathbb{N}^2_0 : 0 \leq n \leq m < \infty\}, \]
\[ H_2 = \{(x, y, m, n) \in \mathbb{N}^4_0 : 0 \leq m \leq x < \infty, 0 \leq n \leq y < \infty\}. \]

Let \( C(M, N) \) denotes the class of continuous functions from \( M \) to \( N \). We use the usual convention that the empty sum is taken to be 0. Throughout, all the functions are assumed to be real-valued and all the integrals and sums involved exist and finite on the respective domains of their definitions.

Our main results are established in the following theorem.

**Theorem 1.** Let \( u(t), a(t) \in C(R_+, R_+), k(t, s), D_1 k(t, s) \in C(G_1, R_+) \) and \( c \) be a nonnegative constant.

(a) Let \( g(u) \) be a continuous, nondecreasing function defined on \( R_+ \) and \( g(u) > 0 \) on \( (0, \infty) \). If
\[
\int_0^t k(t, \sigma) g(u(\sigma)) d\sigma,
\]
for \( t \in R_+ \), then for \( 0 \leq t \leq t_1, t, t_1 \in R_+ \),
\[
u(t) \leq G^{-1}\left[G(c) + \int_0^t A(s) ds\right],
\]
where
\[
A(t) = k(t, t) + \int_0^t D_1 k(t, \sigma) d\sigma,
\]
\[
G(r) = \int_{r_0}^{r} \frac{ds}{g(s)}, \quad r > 0,
\]
\( r_0 > 0 \) is arbitrary, and \( G^{-1} \) is the inverse of \( G \) and \( t_1 \in R_+ \) is chosen so that
\[
G(c) + \int_0^t A(s) ds \in \text{Dom}(G^{-1}),
\]
for all \( t \in R_+ \) lying in the interval \( 0 \leq t \leq t_1 \).
Let $g(u)$ be as in $(a_1)$ and suppose in addition, it is subadditive. If
\[ u(t) \leq a(t) + \int_0^t k(t, \sigma)g(u(\sigma))d\sigma, \]
for $t \in R_+$, then for $0 \leq t \leq t_2$, $t, t_2 \in R_+$,
\[ u(t) \leq a(t) + G^{-1}\left[ G(B(t)) + \int_0^t A(s)ds \right], \]
(2.6)
where $G, G^{-1}, A$ be as in $(a_1)$,
\[ B(t) = \int_0^t k(t, \sigma)g(a(\sigma))d\sigma, \]
and $t_2 \in R_+$ is chosen so that
\[ G(B(t)) + \int_0^t A(s)ds \in \text{Dom}(G^{-1}), \]
for all $t \in R_+$ lying in the interval $0 \leq t \leq t_2$.

The next theorem deals with the two independent variable versions of the
inequalities established in Theorem 1.

**Theorem 2.** Let $u(x, y), a(x, y) \in C(R_+^2, R_+)$, $k(x, y, s, t)$, $D_1k(x, y, s, t)$, $D_2k(x, y, s, t)$, $D_1D_2k(x, y, s, t) \in C(G_2, R_+)$ and $c$ be a nonnegative constant.

(b1) Let $g(u)$ be a continuously differentiable function defined for $u \geq 0$, $g(u) > 0$ for $u > 0$ and $g'(u) \geq 0$ for $u \geq 0$. Let $G, G^{-1}$ be as in $(a_1)$. If
\[ u(x, y) \leq c + \int_0^x \int_0^y k(x, y, s, t)g(u(s, t))dtds, \]
for $x, y \in R_+$, then for $0 \leq x \leq x_1$, $0 \leq y \leq y_1$, $x, x_1, y, y_1 \in R_+$,
\[ u(x, y) \leq G^{-1}\left[ G(c) + \int_0^x \int_0^y P(s, t)dtds \right], \]
(2.9)
where
\[ P(x, y) = k(x, y, x, y) + \int_0^x D_1k(x, y, \sigma, y)d\sigma \\
+ \int_0^y D_2k(x, y, x, \tau)d\tau + \int_0^x \int_0^y D_1D_2k(x, y, \sigma, \tau)d\tau d\sigma, \]
(2.10)
for $x, y \in R_+$ and $x_1, y_1 \in R_+$ are chosen so that
\[ G(c) + \int_0^x \int_0^y P(s, t)dtds \in \text{Dom}(G^{-1}), \]
for all $x, y$ lying in $[0, x_1], [0, y_1]$ respectively.
Let \( g \) be as in \((b_1)\) and suppose in addition, it is subadditive. Let \( G, G^{-1} \) be as in \((a_1)\). If
\[
\begin{align*}
  u(x, y) &\leq a(x, y) + \int_0^x \int_0^y k(x, y, s, t) g(u(s, t)) dt ds,
\end{align*}
\]
for \( x, y \in \mathbb{R}_+ \), then for \( 0 \leq x \leq x_2, 0 \leq y \leq y_2, x, x_2, y, y_2 \in \mathbb{R}_+ \),
\[
\begin{align*}
  u(x, y) &\leq a(x, y) + G^{-1} \left[ G(E(x, y)) + \int_0^x \int_0^y P(s, t) dt ds \right],
\end{align*}
\]
where \( P(x, y) \) is defined by \((2.10)\),
\[
\begin{align*}
  E(x, y) &= \int_0^x \int_0^y k(x, y, s, t) g(a(s, t)) dt ds,
\end{align*}
\]
for \( x, y \in \mathbb{R}_+ \) and \( x_2, y_2 \in \mathbb{R}_+ \) are chosen so that
\[
\begin{align*}
  G(E(x, y)) + \int_0^x \int_0^y P(s, t) dt ds &\in \text{Dom}(G^{-1}),
\end{align*}
\]
for all \( x, y \) lying in \([0, x_2], [0, y_2] \) respectively.

The discrete analogues of the inequalities in Theorems 1 and 2 are given in the following theorems.

**Theorem 3.** Let \( u(n), a(n) \) be nonnegative functions defined on \( N_0 \), \( k(n, s) \), \( \Delta_1 k(n, s) \) be nonnegative functions defined on \( H_1 \), and \( c \) be a nonnegative constant.

\((c_1)\) Let \( g, G, G^{-1} \) be as in \((a_1)\). If
\[
\begin{align*}
  u(n) &\leq c + \sum_{\sigma=0}^{n-1} k(n, \sigma) g(u(\sigma)),
\end{align*}
\]
for \( n \in N_0 \), then for \( 0 \leq n \leq n_1, n, n_1 \in N_0 \),
\[
\begin{align*}
  u(n) &\leq G^{-1} \left[ G(c) + \sum_{s=0}^{n-1} H(s) \right],
\end{align*}
\]
where
\[
\begin{align*}
  H(n) &= k(n + 1, n) + \sum_{\sigma=0}^{n-1} \Delta_1 k(n, \sigma),
\end{align*}
\]
for \( n \in N_0 \) and \( n_1 \in N_0 \) is chosen so that
\[
\begin{align*}
  G(c) + \sum_{s=0}^{n-1} H(s) &\in \text{Dom}(G^{-1}),
\end{align*}
\]
for all \( n \in N_0 \) lying in \( 0 \leq n \leq n_1 \).
(c2) Let \( g, G, G^{-1} \) be as in (a2). If

\[
u(n) \leq a(n) + \sum_{s=0}^{n-1} k(n, \sigma) g(u(\sigma)),
\]

(2.17)

for \( n \in N_0 \), then for \( 0 \leq n \leq n_2 \), \( n, n_2 \in N_0 \),

\[
u(n) \leq a(n) + G^{-1} \left[ G(B(n)) + \sum_{s=0}^{n-1} H(s) \right],
\]

(2.18)

where \( H(n) \) is defined by (2.16),

\[
B(n) = \sum_{\sigma=0}^{n-1} k(n, \sigma) g(a(\sigma)),
\]

(2.19)

and \( n \in N_0 \) and \( n_2 \in N_0 \) is chosen so that

\[
G(B(n)) + \sum_{s=0}^{n-1} H(s) \in \text{Dom}(G^{-1}),
\]

(2.20)

for all \( n \in N_0 \) lying in \( 0 \leq n \leq n_2 \).

**Theorem 4.** Let \( u(x, y), a(x, y) \) be nonnegative functions defined for \( x, y \in N_0 \), \( k(x, y, s, t), \Delta_1 k(x, y, s, t), \Delta_2 k(x, y, s, t), \Delta_2 \Delta_1 k(x, y, s, t) \) be nonnegative functions defined on \( H_2 \) and \( c \) be a nonnegative constant.

(d1) Let \( g, G, G^{-1} \) be as in (a1). If

\[
u(x, y) \leq c + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} k(x, y, s, t) g(u(s, t)),
\]

(2.20)

for \( x, y \in N_0 \), then for \( 0 \leq x \leq x_1 \), \( 0 \leq y \leq y_1 \), \( x, x_1, y, y_1 \in N_0 \),

\[
u(x, y) \leq G^{-1} \left[ G(c) + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} Q(s, t) \right],
\]

(2.21)

where

\[
Q(x, y) = k(x + 1, y + 1, x, y) + \sum_{\sigma=0}^{x-1} \Delta_1 k(x, y + 1, \sigma, y)
\]

\[ + \sum_{\tau=0}^{y-1} \Delta_2 k(x + 1, y, x, \tau) + \sum_{\sigma=0}^{x-1} \sum_{\tau=0}^{y-1} \Delta_1 \Delta_2 k(x, y, \sigma, \tau),
\]

(2.22)
for \( x, y \in N_0 \) and \( x_1, y_1 \in N_0 \) are chosen so that
\[
G(c) + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} Q(s, t) \in \text{Dom}(G^{-1}),
\]
for all \( x \) and \( y \) lying in \( 0 \leq x \leq x_1 \) and \( 0 \leq y \leq y_1 \).

(d2) Let \( g, G, G^{-1} \) be as in (a2). If
\[
u(x, y) \leq a(x, y) + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} k(x, y, s, t)g(u(s, t)),
\]
for \( x, y \in N_0 \), then for \( 0 \leq x \leq x_2 \), \( 0 \leq y \leq y_2 \), \( x, x_2, y, y_2 \in N_0 \),
\[
u(x, y) \leq a(x, y) + G^{-1} \left[ G(\bar{E}(x, y)) + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} Q(s, t) \right],
\]
for \( x, y \in N_0 \), where \( Q(x, y) \) is defined (2.22),
\[
\bar{E}(x, y) = \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} k(x, y, s, t)g(a(s, t)),
\]
for \( x, y \in N_0 \) and \( x_2, y_2 \in N_0 \) are chosen so that
\[
G(\bar{E}(x, y)) + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} Q(s, t) \in \text{Dom}(G^{-1}),
\]
for all \( x \) and \( y \) lying in \( 0 \leq x \leq x_2 \) and \( 0 \leq y \leq y_2 \).

3. Proofs of Theorems 1–4

Since the proofs resemble one another, we give the details for (a1), (a2),
(b1) and (c1) only; the proofs of the remaining inequalities can be completed by
following the proofs of the above mentioned inequalities and closely looking at
the proofs of the similar results given in [4, 5, 6].

(a1) We first assume that \( c > 0 \) and define a function \( z(t) \) by the right hand
side of (2.1). Then \( z(0) = c, u(t) \leq z(t) \) and
\[
z'(t) = k(t, t)g(u(t)) + \int_0^t D_1k(t, \sigma)g(u(\sigma))d\sigma
\leq k(t, t)g(z(t)) + \int_0^t D_1k(t, \sigma)g(z(\sigma))d\sigma
\leq A(t)g(z(t)),
\]
(3.1)
where \( A(t) \) is defined by (2.3). From (2.4) and (3.1) we have
\[
\frac{d}{dt} G(z(t)) = \frac{z'(t)}{g(z(t))} \leq A(t). \tag{3.2}
\]
Now by setting \( t = s \) in (3.2) and integrating it from 0 to \( t \) we have
\[
G(z(t)) \leq G(c) + \int_0^t A(s) ds. \tag{3.3}
\]
Since \( G^{-1} \) is increasing, from (3.3) we have
\[
z(t) \leq G^{-1} \left[ G(c) + \int_0^t A(s) ds \right]. \tag{3.4}
\]
Using (3.4) in \( u(t) \leq z(t) \) gives the required inequality in (2.2).

If \( c \) is nonnegative, we carry out the above procedure with \( c + \epsilon \) instead of \( c \), where \( \epsilon > 0 \) is an arbitrary small constant, and subsequently pass to the limit \( \epsilon \to 0 \) to obtain (2.2). The subinterval \( 0 \leq t \leq t_1 \) is obvious.

(a2) Define a function \( z(t) \) by
\[
z(t) = \int_0^t k(t, \sigma) g(u(\sigma)) d\sigma. \tag{3.5}
\]
Then \( z(0) = 0 \) and from (2.5) we have
\[
u(t) \leq a(t) + z(t). \tag{3.6}
\]
Using (3.6) in (3.5) we have
\[
z(t) \leq \int_0^t k(t, \sigma) g(a(\sigma) + z(\sigma)) d\sigma
\leq B(t) + \int_0^t k(t, \sigma) g(z(\sigma)) d\sigma,
\]
where \( B(t) \) is defined by (2.7). It is easy to observe that \( B(t) \) is nonnegative and nondecreasing in \( t \). Now by following the similar arguments as in the proof of Theorem 2.4.2 given in [4] we get
\[
z(t) \leq G^{-1} \left[ G(B(t)) + \int_0^t A(s) ds \right]. \tag{3.7}
\]
Using (3.7) in (3.6) we get the desired inequality in (2.6).
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(b1) We note that since \( g'(u) \geq 0 \) on \( R_+ \), the function \( g(u) \) is monotonically increasing on \((0, \infty)\). Let \( c > 0 \) and define a function \( z(x, y) \) by the right hand side of (2.8). Then \( z(0, y) = z(x, 0) = c, u(x, y) \leq z(x, y) \) and

\[
D_1 D_2 z(x, y) = k(x, y, x, y)g(u(x, y)) + \int_0^x D_1 k(x, y, \sigma, y)g(u(\sigma, y))d\sigma \\
+ \int_0^y D_2 k(x, y, x, \tau)g(u(x, \tau))d\tau + \int_0^x \int_0^y D_1 D_2 k(x, y, \sigma, \tau)g(u(\sigma, \tau))d\tau d\sigma \\
\leq P(x, y)g(z(x, y)),
\]

where \( P(x, y) \) is defined by (2.10). Now by following the proof of Theorem 5.2.1 given in [4] with suitable modifications we get

\[
z(x, y) \leq G^{-1} \left[ G(c) + \int_0^x \int_0^y P(s, t)dt ds \right].
\]

Using (3.9) in \( u(x, y) \leq z(x, y) \), we obtain the desired inequality in (2.9). If \( c \) is nonnegative, the proof can be completed as mentioned in the proof of (a1).

(c1) First we assume that \( c > 0 \) and define a function \( z(n) \) by the right hand side of (2.14). Then \( z(0) = c, u(n) \leq z(n) \) and (see [5, p.22])

\[
\Delta z(n) = k(n + 1, n)g(u(n)) + \sum_{\sigma=0}^{n-1} \Delta_1 k(n, \sigma)g(u(\sigma)) \\
\leq H(n)g(z(n)),
\]

where \( H(n) \) is defined by (2.16). The rest of the proof can be completed by following the similar arguments as in the proof of Theorem 2.3.1 given in [5].

4. Applications

In this section, we present applications of the inequality \((b_2)\) in Theorem 2 which provide estimates on the solutions of Volterra integral equation of the form

\[
u(x, y) = h(x, y) + \int_0^x \int_0^y F(x, y, s, t, u(s, t))dt ds,
\]

where \( h \in C(R_+^2, R) \) and \( F \in C(G_2 \times R, R) \).

The following theorem deals with the estimate on the solution of (4.1).
Theorem 5. Suppose that
\[
|h(x, y)| \leq a(x, y), \quad (4.2)
\]
\[
|F(x, y, s, t, u)| \leq k(x, y, s, t)g(|u|), \quad (4.3)
\]
where \(a, k, g\) are as in Theorem 2 part (b2). If \(u(x, y), x, y \in R_+\) is any solution of (4.1), then
\[
|u(x, y)| \leq a(x, y) + G^{-1}\left[G(E(x, y)) + \int_0^x \int_0^y P(s, t)dt\right].
\]
(4.4)
where \(E(x, y)\) and \(P(x, y)\) are defined by (2.13) and (2.10) respectively and \(G, G^{-1}\) are as in Theorem 2 part (b2).

Proof. Let \(u(x, y) \in C(R^2_+, R)\) be a solution of (4.1). Using (4.2), (4.3) in (4.1) we have
\[
|u(x, y)| \leq a(x, y) + \int_0^x \int_0^y k(x, y, s, t)g(|u(s, t)|)dt.
\]
(4.5)
Now an application of the inequality (b2) in Theorem 2 to (4.5) yields the desired estimate in (4.4).

In the following theorem we obtain estimate on the solution of (4.1) by assuming that the function \(F\) satisfies the Lipschitz type condition.

Theorem 6. Suppose that
\[
|F(x, y, s, t, z) - F(x, y, s, t, \bar{z})| \leq k(x, y, s, t)g(|z - \bar{z}|), \quad (4.6)
\]
where \(k, g\) are as in Theorem 2 part (b2). If \(u(x, y), x, y \in R_+\) is a solution of (4.1), then
\[
|u(x, y) - h(x, y)| \leq A(x, y) + G^{-1}\left[G(E_0(x, y)) + \int_0^x \int_0^y P(s, t)dt\right],
\]
(4.7)
for \(x, y \in R_+\), where \(G, G^{-1}\) and \(P\) are as in Theorem 2 part (b2), \(E_0(x, y)\) is defined by the right hand side of (2.13) when \(a(x, y)\) is replaced by \(A(x, y)\) given by
\[
A(x, y) = \int_0^x \int_0^y |F(x, y, \sigma, \tau, h(\sigma, \tau))|d\tau d\sigma,
\]
(4.8)
for \(x, y \in R_+\).

**Proof.** Let \(u(x, y) \in C(R^2_+, R)\) be a solution of (4.1). From (4.1) and (4.6) we observe that

\[
|u(x, y) - h(x, y)| = \left| \int_0^x \int_0^y \left( F(x, y, s, t, u(s, t)) - F(x, y, s, t, h(s, t)) + F(x, y, s, t, h(s, t)) \right) dt ds \right|
\]

\[
\leq A(x, y) + \int_0^x \int_0^y k(x, y, s, t) g(|u(s, t) - h(s, t)|) dt ds. \tag{4.9}
\]

Now a suitable application of the inequality (b2) in Theorem 2 to (4.9) yields (4.7).

We note that the inequality (d2) in Theorem 4 can be used to establish theorems similar to Theorems 5 and 6 for the sum — difference equation of the form

\[
u(x, y) = h(x, y) + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} F(x, y, s, t, u(s, t)), \tag{4.10}
\]

under some suitable conditions on the functions involved in (4.10). For various other applications of the inequalities similar to that of given here, see [4, 5].

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