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# ON THE STABILITY OF LOCAL MINIMA IN MATHEMATICAL PROGRAMMING INVOLVING CONE-CONSTRAINTS

BY

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**Abstract**. In this paper, we establish conditions ensuring that the variation of the local minimum of a mathematical programming problem involving cone-constraints is of the same order of magnitude as the variation in the data. At the same time, we also show that the relative variation of Lagrange multipliers to the variation in the local minimum and in the data remains bounded.

## 1. Introduction

Let $f_0$ be a real-valued function defined on $\mathbb{R}^n$, and let $g_0$ and $h_0$ be mappings from $\mathbb{R}^n$ into $\mathbb{R}^m$ and $\mathbb{R}^s$, respectively. Let $S$ be a closed convex cone in $\mathbb{R}^m$ with nonempty interior. Let us consider the following cone-constrained problem:

$$(P) \qquad \begin{cases} \text{minimize } f_0(x), \\ \text{subject to} \\ \quad -g_0(x) \in S, \\ \quad h_0(x) = 0. \end{cases}$$

For each parameter $\alpha \in \mathbb{R}^p$, we also consider the following perturbed problem for

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(P):

$$(P_\alpha) \qquad \begin{cases} \text{minimize } f(x,\alpha), \\ \text{subject to} \\ \quad -g(x,\alpha) \in S, \\ \quad h(x,\alpha) = 0, \end{cases}$$

where $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$, $g : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^m$, $h : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^s$, and

$$f(x,0) = f_0(x), \quad g(x,0) = g_0(x), \quad h(x,0) = h_0(x).$$

Let $\overline{x}$ and $\overline{x}_\alpha$ be local minima of Problems (P) and (P$_\alpha$), respectively.

The stability of solutions for the problem of this type has been studied by many authors (see, e.g., [1], [3]-[8], [11]-[13], [15], [17], [20], [21], [23], [24]). Perturbation analyses to compute estimates of orders of magnitude of the variation of solutions with respect to perturbations in the data were studied by A. V. Fiacco [12], S. M. Robinson [20], [21], J. Jittorntrum [17], B. Cornet - G. Laroque [6], J. F. Bonnans [3], [4], [5], .... A question arises as to when $\|\overline{x}_\alpha - \overline{x}\|$ is of the order of magnitude of $\|\alpha\|$, which means that when $\|\overline{x}_\alpha - \overline{x}\|/\|\alpha\|$ is bounded as $\alpha \to 0$. This has directly concerned with directional differentiability and Lipschitz continuity of $\overline{x}_\alpha$ (see, e.g., [5], [6]). In case $S$ is the nonnegative orthant $\mathbb{R}^m_+$ in $\mathbb{R}^m$, Bonnans [4] gave an estimate of the variation in Lagrange multipliers related to the variation of solutions and data. He derived a condition ensuring that the variation of solutions is of the same order of magnitude as the variation in data.

Motivated by the results due to Bonnans [4], we study the problem (P) and its perturbed problem (P$_\alpha$) with cone-constraints. In Section 2, we show that if $\overline{x}$ is a local minimizer for (P) and the Mangasarian-Fromovitz constraint qualification holds at $\overline{x}$, then the set of Lagrange multipliers of (P) at $\overline{x}$ is convex, compact, and the local minimizer of (P$_\alpha$) is still a regular point in the sense of Zowe-Kurcyusz [24] after a small perturbation. Section 3 is devoted to the discussion of the relative variation of Lagrange multipliers of (P$_\alpha$) to the variation in solutions and data. Section 4 presents conditions ensuring that $\|\overline{x}_\alpha - \overline{x}\|$ is of the same order of magnitude as $\|\alpha\|$. From these results we obtain those due to Bonnans [4] as a special case.

## 2. Perturbation of a Regular Solution

We now consider Problem (P) and its perturbed problem ($\mathrm{P}_\alpha$). Let $\overline{x}$ and $\overline{x}_\alpha$ be local minimizers for (P) and ($\mathrm{P}_\alpha$), respectively, and $\overline{x}_\alpha \to \overline{x}$. Assume that the functions $f_0$, $g_0$ and $h_0$ are Fréchet differentiable at $\overline{x}$ with the Fréchet derivatives $f_0'(\overline{x})$, $g_0'(\overline{x})$ and $h_0'(\overline{x})$. In what follows we shall show that if $\overline{x}$ satisfies the regularity condition of Mangasarian-Fromovitz type, then $\overline{x}_\alpha$ is a regular point in the sense of Zowe-Kurcyusz [24].

**Definition 2.1.**(Kawasaki [18]) The system

$$-g(x) \in S, \quad h(x) = 0, \tag{1}$$

is said to satisfy the Mangasarian-Fromovitz condition at $\overline{x}$ if there exists $d_0 \in \mathbb{R}^n$ such that

(i) $h'(\overline{x})d_0 = 0$;
(ii) $g(\overline{x}) + g'(\overline{x})d_0 \in -\mathrm{int}\, S$;
(iii) $h'(\overline{x})$ is a surjective.

Note that in the special case $S = \mathbb{R}^m_+$, conditions (i)-(iii) are reduced to the constraint qualification due to Mangasarian-Fromovitz [19].

We set $G = (g, h)$, $S_1 = S \times \{O_s\}$, where $O_s$ is the origin of $\mathbb{R}^s$. Then $S_1$ is a closed convex cone in $\mathbb{R}^{m+s}$. Denote by cone $A$ the cone generated by a set $A$. For $c \in S_1$, let

$$S_c = \mathrm{cone}\,(S_1 - c).$$

Then $S_c$ is a convex cone, not always closed. Thus

$$S_{G(\overline{x})} = \mathrm{cone}\,(S \times \{O_s\} + G(\overline{x})).$$

Observing that $G(\overline{x}) \in -S_1$, from [2, p. 50] one gets

$$S_{G(\overline{x})} = S_1 + \{\lambda G(\overline{x}) : \lambda \geq 0\}.$$

Denote by $S_1^*$ the dual cone of $S_1$

$$S_1^* = \{\xi \in \mathbb{R}^{m+s} : \langle \xi, x\rangle \geq 0 \quad \text{for all } x \in S_1\}.$$

It should be noted here that $\xi \in S^*_{G(\overline{x})}$ if and only if $\xi \in S_1^*$ and $\langle \xi, G(\overline{x})\rangle = 0$.

**Definition 2.2.**(Zowe-Kurcyusz [24]) $\overline{x}$ is called a regular point in the sense of Zowe-Kurcyusz, if

$$G'(\overline{x})\mathbb{R}^n + S_{G(\overline{x})} = \mathbb{R}^{m+s}. \tag{2}$$

Adapting the definition due to Zowe-Kurcyusz [24], a vector $(\lambda, \mu) \in \mathbb{R}^m \times \mathbb{R}^s$ is called a Lagrange multiplier (or Kuhn-Tucker vector) for (P) at the optimal $\overline{x}$, if $\lambda \in S^*$ and

$$f'(\overline{x}) + g'(\overline{x})\lambda + h'(\overline{x})\mu = 0,$$
$$\langle \lambda, g(\overline{x})\rangle = 0,$$

Denote by $\Lambda(\overline{x})$ the set of Lagrange multipliers for (P) at $\overline{x}$.

**Theorem 2.1.** *Let $\overline{x}$ be a local minimizer of Problem* (P)*, and let $g'_x(\cdot,\cdot)$ and $h'_x(\cdot,\cdot)$ be continuous at $(\overline{x}, 0)$. Assume that the Mangasarian-Fromovitz constraint qualification is fulfilled at $\overline{x}$ for* (P)*. Then, the following two conclusions hold:*

(i) $\Lambda(\overline{x})$ *is nonempty, convex and compact;*

(ii) *for $\alpha \in \mathbb{R}^p$ such that $\|\alpha\|$ is sufficiently small, $\overline{x}_\alpha$ is a regular point in the sense of Zowe-Kurcyusz for* $(\mathrm{P}_\alpha)$.

**Proof.** We first begin with showing that $\overline{x}$ is a regular point in the sense of Zowe-Kurcyucz, i.e.

$$G'(\overline{x})\mathbb{R}^n + S_{G(\overline{x})} = \mathbb{R}^{m+s}. \tag{3}$$

Assume the contrary, that

$$G'(\overline{x})\mathbb{R}^n + S_{G(\overline{x})} \subsetneqq \mathbb{R}^{m+s}.$$

This leads to the existence of a point $(u_1, u_2) \in \mathbb{R}^m \times \mathbb{R}^s \setminus [G'(\overline{x})\mathbb{R}^n + S_{G(\overline{x})}]$. Since $S_{G(\overline{x})}$ is a convex cone containing the origin, $G'(\overline{x})\mathbb{R}^n + S_{G(\overline{x})}$ is a nonempty convex cone in $\mathbb{R}^{m+s}$.

Making use of a separation theorem for disjoint convex sets $\{(u_1, u_2)\}$ and $G'(\overline{x})\mathbb{R}^n + S_{G(\overline{x})}$ in the finite dimensional space $\mathbb{R}^{m+s}$ (see, e.g., [22, Theorem 11.3]), we claim that there exists $(\lambda^*, \mu^*) \in \mathbb{R}^m \times \mathbb{R}^s \setminus \{0\}$ such that

$$\langle \lambda^*, u_1\rangle + \langle \mu^*, u_2\rangle \le \langle \lambda^*, y\rangle + \langle \mu^*, z\rangle \quad (\forall (y, z) \in G'(\overline{x})\mathbb{R}^n + S_{G(\overline{x})}).$$

Since $G'(\overline{x})\mathbb{R}^n + S_{G(\overline{x})}$ is a convex cone, it follows from Lemma 5.1 in [14] that

$$\langle \lambda^*, y\rangle + \langle \mu^*, z\rangle \geq 0 \quad (\forall (y,z) \in G'(\overline{x})\mathbb{R}^n + S_{G(\overline{x})}),$$

which implies that

$$\langle \lambda^*, g'(\overline{x})d\rangle + \langle \mu^*, h'(\overline{x})d\rangle \geq 0 \quad (\forall\, d \in \mathbb{R}^n), \tag{4}$$

$$\langle \lambda^*, y\rangle + \langle \mu^*, z\rangle \geq 0 \quad (\forall (y,z) \in S_{G(\overline{x})}), \tag{5}$$

as $S_{G(\overline{x})}$ contains the origin.

It follows from (5) that $(\lambda^*, \mu^*) \in S^*_{G(\overline{x})}$. This is equivalent to the following

$$\lambda^* \in S^*, \quad \mu^* \in \mathbb{R}^s, \quad \langle \lambda^*, g(\overline{x})\rangle = 0. \tag{6}$$

We shall show that $\lambda^* \neq 0$.

If this were not so, i.e., $\lambda^* = 0$, then there would be $\mu^* \neq 0$, as $(\lambda^*, \mu^*) \neq 0$, and (4) would become

$$\langle \mu^*, h'(\overline{x})d\rangle \geq 0 \quad (\forall\, d \in \mathbb{R}^n).$$

Since $h'(\overline{x})$ is surjective from $\mathbb{R}^n$ onto $\mathbb{R}^s$, it follows that

$$\langle \mu^*, z\rangle = 0 \quad (\forall\, z \in \mathbb{R}^s),$$

which implies that $\mu^* = 0$. This conflicts with $\mu^* \neq 0$, and hence $\lambda^* \neq 0$.

On the other hand, for $\lambda^* \in S^* \setminus \{0\}$ and $\mu^* \in \mathbb{R}^s$, in view of assumptions (i) and (ii), we obtain

$$\langle \lambda^*, g(\overline{x}) + g'(\overline{x})d_0\rangle < 0,$$
$$\langle \mu^*, h'(\overline{x})d_0\rangle = 0,$$

which leads to the following

$$\langle \lambda^*, g(\overline{x}) + g'(\overline{x})d_0\rangle + \langle \mu^*, h'(\overline{x})d_0\rangle < 0.$$

This along with (6) yields that

$$\langle \lambda^*, g'(\overline{x})d_0\rangle + \langle \mu^*, h'(\overline{x})d_0\rangle < 0.$$

But this contradicts (4), and hence, (3) holds. Thus $\overline{x}$ is a regular point in the sense of Zowe-Kurcyusz.

Theorem 3.1 in [21] claimed that the set of Lagrange multipliers at an optimal solution, which is regular in the sense of Zowe-Kurcyusz, is nonempty, while Theorem 4.1 and Remark 4.(i) in [24] also contended that this set is convex and weakly compact. Hence, $\Lambda(\overline{x})$ here is nonempty, convex and compact, as the considering problem is in finite dimentional spaces. By hypotheses, $g'_x(\cdot,\cdot)$ and $h'_x(\cdot,\cdot)$ are continuous at $(\overline{x},0)$, $g'_x(\overline{x}_\alpha,\alpha) \to g'_x(\overline{x},0) = g'_0(\overline{x})$ and $h'_x(\overline{x}_\alpha,\alpha) \to h'_x(\overline{x},0) = h'_0(\overline{x})$ as $\alpha \to 0$. On the other hand, Theorem 5.2 in [24] infered that if the local minimizer of the original problem is regular in the sense of Zowe-Kurcyusz, the local minimizer of perturbed problem is still a Zowe-Kurcyusz regular point after a small perturbation. In the above we have just shown that $\overline{x}$ is a Zowe-Kurcyusz regular point. Hence, for $\alpha$ small enough, $\overline{x}_\alpha$ is a Zowe-Kurcyusz regular point as well. This concludes the proof.

**Discussion 2.1.** In the paper, we make the assumption that the perturbed problem reaches a minimizer at a point $\overline{x}_\alpha$, which tends to the minimizer $\overline{x}$ of the unperturbed problem as $\alpha \to 0$. Under this assumption it can be obtained the results on stability of Lagrange multipliers as in this paper and Bonnans [4], those on Lipschitz continuity of $\overline{x}_\alpha$ as in Craven [8], Craven-Luu [9], [10], Cornet-Laroque [6], those on directional differentiability of $\overline{x}_\alpha$ as in Bonnans [5], Gauvin-Janin [13], and those on estimating the distance between the local minimizers of perturbed problems and the local minimizer of the original problem as in this paper and Bonnans [4], Auslender [1]. The assumption aforementioned holds if such as $\overline{x}$ is a strict minimum as in Craven [7], [8], Craven-Luu [11], or an isolated local minimum as in Auslender [1], Studniarski [23], Hoa-Luu [15].

## 3. Relative Variation of Lagrange Multipliers

Let $\overline{x}$ and $\overline{x}_\alpha$ be local minimizers of Problems (P) and (P$_\alpha$), respectively, and $\overline{x}_\alpha \to \overline{x}$. Denote by $\Lambda(x_\alpha)$ the set of Lagrange multipliers of (P$_\alpha$) at $\overline{x}_\alpha$. When the Mangasarian-Fromovitz condition holds at $\overline{x}$, due to Theorem 2.1 $\overline{x}_\alpha$ is a regular point in the sense of Zowe-Kurcyusz for $\alpha$ small enough. On using Theorem 3.1 in [24] yields that $\Lambda(\overline{x}_\alpha)$ is nonempty for $\alpha$ small enough.

Given a subset $A$ of $\mathbb{R}^{m+s}$, we denote by $d_A$ the distance function of $A$:

$$d_A(x) = \inf\{\|x-a\| : a \in A\}.$$

For $(\lambda_\alpha, \mu_\alpha) \in \Lambda(\overline{x}_\alpha)$, by virtue of Theorem 2.1, $\Lambda(\overline{x})$ is compact, and hence,

$$d_{\Lambda(\overline{x})}(\lambda_\alpha, \mu_\alpha) = \min\{\|(\lambda_\alpha, \mu_\alpha) - (\lambda, \mu)\| : (\lambda, \mu) \in \Lambda(\overline{x})\}, \tag{7}$$

where $\|(\lambda_\alpha, \mu_\alpha) - (\lambda, \mu)\| = [(\lambda_\alpha - \lambda)^2 + (\mu_\alpha - \mu)^2]^{1/2}$.

Recall [4] that the relative variation of Lagrange multipliers to the variation in solutions and data is defined as

$$r_\alpha = \frac{d_{\Lambda(\overline{x})}(\lambda_\alpha, \mu_\alpha)}{\|\overline{x}_\alpha - \overline{x}\| + \|\alpha\|}\,.$$

The following theorem shows that $r_\alpha$ remains bounded as $\alpha \to 0$.

**Theorem 3.1.** *Assume that the function $(f(\cdot,\cdot), g(\cdot,\cdot), h(\cdot,\cdot))$ belongs to the class $C^2$ and the Mangasarian-Fromovitz condition holds at $\overline{x}$. Then $r_\alpha$ is bounded as $\alpha \to 0$.*

**Proof.** Assume the contrary, that $r_\alpha \to \infty$ as $\alpha \to 0$. Then $1/r_\alpha \to 0$ as $\alpha \to 0$. It follows from Theorem 2.1 that for $\alpha \in \mathbb{R}^p$ such that $\|\alpha\|$ is small enough, $\overline{x}_\alpha$ is a regular point in the sense of Zowe-Kurcyusz for $(\mathrm{P}_\alpha)$.

For every $(\lambda_\alpha, \mu_\alpha) \in \Lambda(x_\alpha)$, in view of (7), there exists $(\hat{\lambda}_\alpha, \hat{\mu}_\alpha) \in \Lambda(\overline{x})$ such that

$$\|(\lambda_\alpha, \mu_\alpha) - (\hat{\lambda}_\alpha, \hat{\mu}_\alpha)\| = \min\{\|(\lambda_\alpha, \mu_\alpha) - (\lambda, \mu)\| : (\lambda, \mu) \in \Lambda(\overline{x})\},$$

where $\|(\lambda_\alpha, \mu_\alpha) - (\hat{\lambda}_\alpha, \hat{\mu}_\alpha)\| = ((\lambda_\alpha - \hat{\lambda}_\alpha)^2 + (\mu_\alpha - \hat{\mu}_\alpha)^2)^{1/2}$.

We define the mapping $P : (\lambda_\alpha, \mu_\alpha) \mapsto (\hat{\lambda}_\alpha, \hat{\mu}_\alpha)$. Observing that $P$ is well defined iff $(\hat{\lambda}_\alpha, \hat{\mu}_\alpha)$ is unique, we deduce that it is well defined iff $(\hat{\lambda}_\alpha, \hat{\mu}_\alpha)$ is the unique solution of the following problem

$$\min\{(\lambda - \lambda_\alpha)^2 + (\mu - \mu_\alpha)^2 : (\lambda, \mu) \in \Lambda(\overline{x})\}. \tag{8}$$

$(\hat{\lambda}_\alpha, \hat{\mu}_\alpha)$ is unique, since this is a convex problem in which the objective function is strictly convex and $\Lambda(\overline{x})$ is a convex set. Hence, a necessary and sufficient condition ensuring that $(\hat{\lambda}_\alpha, \hat{\mu}_\alpha)$ is the unique solution of Problem (8) (see, e.g., [16, Theorem 2', p. 77]) is as follows

$$0 \in (2(\hat{\lambda}_\alpha - \lambda_\alpha), 2(\hat{\mu}_\alpha - \mu_\alpha)) + N_{\Lambda(\overline{x})}(\hat{\lambda}_\alpha, \hat{\mu}_\alpha),$$

where $N_{\Lambda(\overline{x})}(\hat{\lambda}_\alpha, \hat{\mu}_\alpha)$ is the normal cone to the convex set $\Lambda(\overline{x})$ at $(\hat{\lambda}_\alpha, \hat{\mu}_\alpha)$. This means that

$$(\lambda_\alpha - \hat{\lambda}_\alpha, \mu_\alpha - \hat{\mu}_\alpha) \in N_{\Lambda(\overline{x})}(\hat{\lambda}_\alpha, \hat{\mu}_\alpha),$$

which can be rewritten equivalently as

$$(\lambda_\alpha - \hat{\lambda}_\alpha)(\lambda - \hat{\lambda}_\alpha) + (\mu_\alpha - \hat{\mu}_\alpha)(\mu - \hat{\mu}_\alpha) \leq 0 \quad (\forall\, (\lambda, \mu) \in \Lambda(\overline{x})). \tag{9}$$

Thus $P$ is well defined if and only if (9) holds.

Observe that for $(\lambda_\alpha, \mu_\alpha) \in \Lambda(\overline{x}_\alpha)$,

$$f'_x(\overline{x}, \alpha) + g'_x(\overline{x}_\alpha, \alpha)\lambda_\alpha + h'_x(\overline{x}_\alpha, \alpha)\mu_\alpha = 0,$$

where $f'_x$, $g'_x$ and $h'_x$ are partial derivatives of $f$, $g$ and $h$ in the first variable, respectively. Taylor's expansion of the function $f'_x(\overline{x}_\alpha, \alpha) + g'_x(\overline{x}_\alpha, \alpha)\lambda_\alpha + h'_x(\overline{x}_\alpha, \alpha)\mu_\alpha$ at $\alpha = 0$, $\overline{x}_\alpha = \overline{x}$, $(\lambda, \mu) \in \Lambda(\overline{x})$ is as follows

$$\begin{aligned} 0 = \alpha \frac{\partial}{\partial \alpha} & [f'_x(\overline{x}_\alpha, \alpha) + g'_x(\overline{x}_\alpha, \alpha)\lambda + h'_x(\overline{x}_\alpha, \alpha)\mu]|_{\alpha=0} \\ & + [f''_0(\overline{x}) + g''_0(\overline{x})\lambda + h''_0(\overline{x})\mu](\overline{x}_\alpha - \overline{x}) + g'_0(\overline{x})(\lambda_\alpha - \lambda) \\ & + h'_0(\overline{x})(\mu_\alpha - \mu) + o(\|(\overline{x}_\alpha - \overline{x})\| + \|\alpha\| + \|(\lambda_\alpha, \mu_\alpha) - (\lambda, \mu)\|), \end{aligned} \tag{10}$$

as $f'_0(\overline{x}) + g'_0(\overline{x})\lambda + h'_0(\overline{x})\mu = 0$.

Put

$$H_{(\lambda,\mu)}(\overline{x}) = f''(\overline{x}) + g''(\overline{x})\lambda + h''(\overline{x})\mu.$$

Taking $(\lambda, \mu) = (\hat{\lambda}_\alpha, \hat{\mu}_\alpha)$, dividing both sides of (10) by $\|(\lambda_\alpha, \mu_\alpha) - (\hat{\lambda}_\alpha, \hat{\mu}_\alpha)\|$ and setting

$$\gamma_\alpha = \frac{(\lambda_\alpha - \hat{\lambda}_\alpha, \mu_\alpha - \hat{\mu}_\alpha)}{\|(\lambda_\alpha, \mu_\alpha) - (\hat{\lambda}_\alpha, \hat{\mu}_\alpha)\|},$$

it follows from (10) that

$$\begin{aligned} & \frac{\alpha}{\|(\lambda_\alpha, \mu_\alpha) - (\hat{\lambda}_\alpha, \hat{\mu}_\alpha)\|} [f''_{x\alpha}(\overline{x}, \alpha) + g''_{x\alpha}(\overline{x}, \alpha)\hat{\lambda}_\alpha + h''_{x\alpha}(\overline{x}, \alpha)\hat{\mu}_\alpha]|_{\alpha=0} \\ & + H_{(\hat{\lambda}_\alpha, \hat{\mu}_\alpha)}(\overline{x}) \Big( \frac{\overline{x}_\alpha - \overline{x}}{\|(\lambda_\alpha, \mu_\alpha) - (\hat{\lambda}_\alpha, \hat{\mu}_\alpha)\|} \Big) + (g'_0(\overline{x}), h'_0(\overline{x}))\gamma_\alpha \\ & + \frac{o(\|\overline{x}_\alpha - \overline{x}\| + \|\alpha\| + \|(\lambda_\alpha, \mu_\alpha) - (\hat{\lambda}_\alpha, \hat{\mu}_\alpha)\|)}{\|(\lambda_\alpha, \mu_\alpha) - (\hat{\lambda}_\alpha, \hat{\mu}_\alpha)\|} = 0. \end{aligned} \tag{11}$$

In view of the compactness of $\Lambda(\overline{x})$, there exists a subsequence $\{(\hat{\lambda}_{\alpha_k}, \hat{\mu}_{\alpha_k})\}$ of $\{(\hat{\lambda}_\alpha, \hat{\mu}_\alpha)\}$ such that $(\hat{\lambda}_{\alpha_k}, \hat{\mu}_{\alpha_k}) \to (\overline{\lambda}, \overline{\mu}) \in \Lambda(\overline{x})$. Observing that $1/r_\alpha \to 0$ as $\alpha \to 0$, we have

$$\frac{\|\overline{x}_\alpha - \overline{x}\| + \|\alpha\|}{\|(\lambda_\alpha, \mu_\alpha) - (\hat{\lambda}_\alpha, \hat{\mu}_\alpha)\|} \to 0 \quad \text{as } \alpha \to 0,$$

which implies that

$$\frac{\|\overline{x}_\alpha - \overline{x}\|}{\|(\lambda_\alpha, \mu_\alpha) - (\hat{\lambda}_\alpha, \hat{\mu}_\alpha)\|} \to 0,$$
$$\frac{\|\alpha\|}{\|(\lambda_\alpha, \mu_\alpha) - (\hat{\lambda}_\alpha, \hat{\mu}_\alpha)\|} \to 0 \quad \text{as } \alpha \to 0.$$

Since $\|\gamma_\alpha\| = 1$ and the unit sphere in $\mathbb{R}^{m+s}$ is compact, without loss of generality, we can assume that $\gamma_{\alpha_k} \to \overline{\gamma} = (\overline{\gamma}_1, \overline{\gamma}_2) \in \mathbb{R}^m \times \mathbb{R}^s$. Therefore, it follows from (11) that

$$(g_0'(\overline{x}), h_0'(\overline{x}))\overline{\gamma} = 0,$$

which means that

$$g_0'(\overline{x})\overline{\gamma}_1 + h_0'(\overline{x})\overline{\gamma}_2 = 0. \tag{12}$$

Since $(\hat{\lambda}_{\alpha_k}, \hat{\mu}_{\alpha_k}) \in \Lambda(\overline{x})$,

$$f_0'(\overline{x}) + g_0'(\overline{x})\hat{\lambda}_{\alpha_k} + h_0'(\overline{x})\hat{\mu}_{\alpha_k} = 0. \tag{13}$$

Combining (12) and (13) yields that for any $\sigma > 0$,

$$f_0'(\overline{x}) + g_0'(\overline{x})(\hat{\lambda}_{\alpha_k} + \sigma\overline{\gamma}_1) + h_0'(\overline{x})(\hat{\mu}_{\alpha_k} + \sigma\overline{\gamma}_2) = 0. \tag{14}$$

Note that $\hat{\mu}_{\alpha_k} + \sigma\overline{\gamma}_2 \in \mathbb{R}^s$ for all $\sigma > 0$.

We now observe that $\gamma_{\alpha_k} = (\gamma^1_{\alpha_k}, \gamma^2_{\alpha_k})$ in which

$$\gamma^1_{\alpha_k} = \frac{\lambda_{\alpha_k} - \hat{\lambda}_{\alpha_k}}{\|(\lambda_{\alpha_k}, \mu_{\alpha_k}) - (\hat{\lambda}_{\alpha_k}, \hat{\mu}_{\alpha_k})\|}, \quad \gamma^2_{\alpha_k} = \frac{\mu_{\alpha_k} - \hat{\mu}_{\alpha_k}}{\|(\lambda_{\alpha_k}, \mu_{\alpha_k}) - (\hat{\lambda}_{\alpha_k}, \hat{\mu}_{\alpha_k})\|}.$$

We shall show that there is a natural number $N$ such that for every $k \geq N$, there exists $\sigma_k > 0$ such that

$$\hat{\lambda}_{\alpha_k} + \sigma_k\overline{\gamma}_1 \in S^*. \tag{15}$$

If it was not so, then for any natural number $N$, there exists $k \geq N$ such that for every $\sigma > 0$, we should have

$$\hat{\lambda}_{\alpha_k} + \sigma\overline{\gamma}_1 \notin S^*.$$

In view of the closedness of $S^*$, $\hat{\lambda}_{\alpha_k} + \sigma\overline{\gamma}_1$ belongs to the open set $\mathbb{R}^m \setminus S^*$ for all $\sigma > 0$. Since $\gamma^1_{\alpha_k} \to \overline{\gamma}_1$, for $k$ large enough,

$$\hat{\lambda}_{\alpha_k} + \sigma\gamma^1_{\alpha_k} \notin S^*. \tag{16}$$

On the other hand,

$$\begin{aligned}\hat{\lambda}_{\alpha_k} + \sigma\gamma^1_{\alpha_k} &= \hat{\lambda}_{\alpha_k} + \frac{\sigma(\lambda_{\alpha_k} - \hat{\lambda}_{\alpha_k})}{\|(\lambda_{\alpha_k}, \mu_{\alpha_k}) - (\hat{\lambda}_{\alpha_k}, \hat{\mu}_{\alpha_k})\|} \\ &= \Big(1 - \frac{\sigma}{\|(\lambda_{\alpha_k}, \mu_{\alpha_k}) - (\hat{\lambda}_{\alpha_k}, \hat{\mu}_{\alpha_k})\|}\Big)\hat{\lambda}_{\alpha_k} + \frac{\sigma}{\|(\lambda_{\alpha_k}, \mu_{\alpha_k}) - (\hat{\lambda}_{\alpha_k}, \hat{\mu}_{\alpha_k})\|}\lambda_{\alpha_k}.\end{aligned}$$

Consequently, for $\sigma \in (0, \|(\lambda_{\alpha_k}, \mu_{\alpha_k}) - (\hat{\lambda}_{\alpha_k}, \hat{\mu}_{\alpha_k})\|)$,

$$\hat{\lambda}_{\alpha_k} + \sigma\gamma^1_{\alpha_k} \in S^*,$$

as $S^*$ is a convex cone. But this contradicts (16), and hence (15) holds.

By virtue of (14) and (15) it follows that for $k$ sufficiently large, there exists $\sigma_k > 0$ such that $(\hat{\lambda}_{\alpha_k} + \sigma_k\overline{\gamma}_1,\ \hat{\mu}_{\alpha_k} + \sigma_k\overline{\gamma}_2) \in \Lambda(\overline{x})$. On using (9) to $(\lambda, \mu) = (\hat{\lambda}_{\alpha_k} + \sigma_k\overline{\gamma}_1,\ \hat{\mu}_{\alpha_k} + \sigma_k\overline{\gamma}_2)$, we deduce that

$$(\lambda_{\alpha_k} - \hat{\lambda}_{\alpha_k})\sigma_k\overline{\gamma}_1 + (\mu_{\alpha_k} - \hat{\mu}_{\alpha_k})\sigma_k\overline{\gamma}_2 \leq 0. \tag{17}$$

Dividing by $\sigma_k\|(\lambda_{\alpha_k}, \mu_{\alpha_k}) - (\hat{\lambda}_{\alpha_k}, \hat{\mu}_{\alpha_k})\|$ both sides of (17), we get

$$\frac{(\lambda_{\alpha_k} - \hat{\lambda}_{\alpha_k})\overline{\gamma}_1}{\|(\lambda_{\alpha_k}, \mu_{\alpha_k}) - (\hat{\lambda}_{\alpha_k}, \hat{\mu}_{\alpha_k})\|} + \frac{(\mu_{\alpha_k} - \hat{\mu}_{\alpha_k})\overline{\gamma}_2}{\|(\lambda_{\alpha_k}, \mu_{\alpha_k}) - (\hat{\lambda}_{\alpha_k}, \hat{\mu}_{\alpha_k})\|} \leq 0.$$

By letting $k \to \infty$, we obtain

$$\|\overline{\gamma}_1\|^2 + \|\overline{\gamma}_2\|^2 \leq 0,$$

which implies that $\overline{\gamma}_1 = \overline{\gamma}_2 = 0$, and hence $\overline{\gamma} = 0$. But this conflicts with $\|\overline{\gamma}\| = 1$. The proof is complete.

**Remark 3.1.** From Theorem 3.1 here we can obtain Theorem 1 due to Bonnans [4] as a special case.

## 4. Stability of Solutions

This section deals with conditions ensuring that if $x_\alpha$ are local minima of Problems ($\mathrm{P}_\alpha$) which converges to the local minimum $\overline{x}$ of Problem (P), then the variation of the solution is of the same order of magnitude as the variation in data.

Throughout this section we suppose that $f_0$, $g_0$, $h_0$ are functions of the class $C^2$, $\overline{x}_\alpha$ is a local minimizer of ($\mathrm{P}_\alpha$) which converges to the local minimizer $\overline{x}$ of Problem (P), and the constraint $g(x,\alpha) \in -S$ of ($\mathrm{P}_\alpha$) is active at $\overline{x}_\alpha$, i.e., $g(\overline{x}_\alpha, \alpha) = 0$ for $\alpha$ small enough. We define the following set:

$$L = \{d \in \mathbb{R}^n : f_0'(\overline{x})d = 0, -g_0'(\overline{x})d \in \mathrm{cone}(S + g(\overline{x})),\ h_0'(\overline{x})d = 0\}.$$

A theorem on perturbation analysis can be stated as follows.

**Theorem 4.1.** *Assume that $\overline{x}$ is a local minimizer of Problem* (P)*, which satisfied the Mangasarian-Fromovitz condition. Suppose, in addition, that $f(\cdot,\cdot)$, $g(\cdot,\cdot)$, $h(\cdot,\cdot)$ are functions of the class $C^2$, and for any $0 \neq d \in L$, there exist no $(\lambda,\mu) \in \Lambda(\overline{x})$ and $(\nu_1,\nu_2) \in \mathbb{R}^m \times \mathbb{R}^s$ such that*

$$H_{(\lambda,\mu)}(\overline{x})d + g_0'(\overline{x})\nu_1 + h_0'(\overline{x})\nu_2 = 0, \tag{18}$$

$$\langle \nu_1, g_0'(\overline{x})d\rangle = 0, \quad \langle \nu_2, h_0'(\overline{x})d\rangle = 0, \tag{19}$$

*where $H_{(\lambda,\mu)}(\overline{x}) = f_0''(\overline{x}) + g_0''(\overline{x})\lambda + h_0''(\overline{x})\mu$. Then $\frac{\|\overline{x}_\alpha - \overline{x}\|}{\|\alpha\|}$ is bounded as $\alpha \to 0$.*

**Proof.** Assume the contrary, that $\frac{\|\overline{x}_\alpha - \overline{x}\|}{\|\alpha\|} \to \infty$ as $\alpha \to 0$. Expanding $g(\overline{x}_\alpha,\alpha)$ and $h(\overline{x}_\alpha,\alpha)$ at $\alpha = 0$, $\overline{x}_\alpha = \overline{x}$, we obtain

$$g(\overline{x}_\alpha,\alpha) = \alpha\frac{\partial}{\partial\alpha}g(\overline{x},\alpha)|_{\alpha=0} + g_0'(\overline{x})(\overline{x}_\alpha - \overline{x}) + o(\|\overline{x}_\alpha - \overline{x}\| + \|\alpha\|), \tag{20}$$

$$h(\overline{x}_\alpha,\alpha) = \alpha\frac{\partial}{\partial\alpha}h(\overline{x},\alpha)|_{\alpha=0} + h_0'(\overline{x})(\overline{x}_\alpha - \overline{x}) + o(\|\overline{x}_\alpha - \overline{x}\| + \|\alpha\|), \tag{21}$$

as $g_0(\overline{x}) = g(\overline{x},0) = \lim_{\alpha\to 0} g(\overline{x}_\alpha,\alpha) = 0$, $h_0(\overline{x}) = h(\overline{x},0) = 0$, $g_x'(\overline{x},0) = g_0'(\overline{x})$ and $h_x'(\overline{x},0) = h_0'(\overline{x})$, where $g_x'$ and $h_x'$ are partial derivatives of $f$ and $g$, respectively.

Dividing both sides of (20) by $\|\overline{x}_\alpha - \overline{x}\|$, we get

$$\frac{\alpha}{\|\overline{x}_\alpha - \overline{x}\|}g_\alpha'(\overline{x},0) + g_0'(\overline{x})\frac{(\overline{x}_\alpha - \overline{x})}{\|\overline{x}_\alpha - \overline{x}\|} + \frac{o(\|\overline{x}_\alpha - \overline{x}\| + \|\alpha\|)}{\|\overline{x}_\alpha - \overline{x}\|} = 0, \tag{22}$$

since $g_0(\overline{x}) = 0$ and $g(\overline{x}_\alpha, \alpha) = 0$ for $\alpha$ small enough.

Observe that $\frac{\alpha}{\|\overline{x}_\alpha - \overline{x}\|} \to 0$ as $\alpha \to 0$, and there is a subsequence of $\left\{\frac{\overline{x}_\alpha - \overline{x}}{\|\overline{x}_\alpha - \overline{x}\|}\right\}$ converging to $d$ with $\|d\| = 1$. Without loss of generality, we can assume that $\frac{\overline{x}_\alpha - \overline{x}}{\|\overline{x}_\alpha - \overline{x}\|} \to d$ as $\alpha \to 0$. Hence, it follows from (22) that

$$g_0'(\overline{x})d = 0. \tag{23}$$

Similarly, we also arrive

$$h_0'(\overline{x})d = 0. \tag{24}$$

Making use of Theorem 2.1 yields that $\overline{x}_\alpha$ is a regular point in the sense of Zowe-Kurcyusz for $\alpha$ small enough. Taking account of Theorem 3.1 in [24], we get $\Lambda(\overline{x}_\alpha)$ is nonempty. We invoke Theorem 3.1 here to deduce that there is $M > 0$ such that for $\alpha$ small enough, $0 \le r_\alpha \le M$, which means that

$$d_{\Lambda(\overline{x})}(\lambda_\alpha, \mu_\alpha) \le M(\|\overline{x}_\alpha - \overline{x}\| + \|\alpha\|).$$

By letting $\alpha \to 0$, it follows readily from this that

$$d_{\Lambda(\overline{x})}(\lambda_\alpha, \mu_\alpha) = \inf\{\|(\lambda_\alpha, \mu_\alpha) - (\lambda, \mu)\| : (\lambda, \mu) \in \Lambda(\overline{x})\} \to 0.$$

In view of the compactness of $\Lambda(\overline{x})$, there exists $(\hat{\lambda}_\alpha, \hat{\mu}_\alpha) \in \Lambda(\overline{x})$ such that

$$\|(\lambda_\alpha, \mu_\alpha) - (\hat{\lambda}_\alpha, \hat{\mu}_\alpha)\| = \min\{\|(\lambda_\alpha, \mu_\alpha) - (\lambda, \mu)\| : (\lambda, \mu) \in \Lambda(\overline{x})\},$$

and there exists a subsequence $\{(\hat{\lambda}_{a_k}, \hat{\mu}_{\alpha_k})\}$ of $\{(\hat{\lambda}_\alpha, \hat{\mu}_\alpha)\}$ such that $\{(\hat{\lambda}_{\alpha_k}, \hat{\mu}_{\alpha_k})\}$ converges to $(\overline{\lambda}, \overline{\mu}) \in \Lambda(\overline{x})$. Hence, $\{(\lambda_{\alpha_k}, \mu_{\alpha_k})\}$ converges also to $(\overline{\lambda}, \overline{\mu})$.

Moreover,

$$f_0'(\overline{x})d + \langle \overline{\lambda}, g_0'(\overline{x})d\rangle + \langle \overline{\mu}, h_0'(\overline{x})d\rangle = 0, \tag{25}$$

as $(\overline{\lambda}, \overline{\mu}) \in \Lambda(\overline{x})$. Substituting (23) and (24) into (25) yields that

$$f_0'(\overline{x})d = 0. \tag{26}$$

Combining (23), (24) and (26) gives that $d \in L$.

Expanding the function $f_x'(\overline{x}_\alpha, \alpha) + g_x'(\overline{x}_\alpha, \alpha)\lambda_\alpha + h_x'(\overline{x}_\alpha, \alpha)\mu_\alpha$ at $\alpha = 0$, $\overline{x}_\alpha = \overline{x}$, $(\lambda, \mu) \in \Lambda(\overline{x})$, we obtain

$$\begin{aligned} 0 &= f_x'(\overline{x}_\alpha, \alpha) + g_x'(\overline{x}_\alpha, \alpha)\lambda_\alpha + h_x'(\overline{x}_\alpha, \alpha)\mu_\alpha \\ &= (f_0'(\overline{x}) + g_0'(\overline{x})\lambda + h_0'(\overline{x})\mu) + \alpha\frac{\partial}{\partial\alpha}[f_x'(\overline{x}, \alpha) + g_x'(\overline{x}, \alpha)\lambda + h_x'(\overline{x}, \alpha)\mu]|_{\alpha=0} \\ &\quad + H_{(\lambda,\mu)}(\overline{x})(\overline{x}_\alpha - \overline{x}) + g_0'(\overline{x})(\lambda_\alpha - \lambda) + h_0'(\overline{x})(\mu_\alpha - \mu) \\ &\quad + o(\|\overline{x}_\alpha - \overline{x}\| + \|(\lambda_\alpha, \mu_\alpha) - (\lambda, \mu)\| + \|\alpha\|), \end{aligned} \tag{27}$$

as $f''_{xx}(\overline{x},0) = f''_0(\overline{x})$, $g''_{xx}(\overline{x},0) = g''_0(\overline{x})$ and $h''_{xx}(\overline{x},0) = h''_0(\overline{x})$.

Dividing both sides of (27) by $\|\overline{x}_\alpha - \overline{x}\|$, and taking $\lambda = \hat{\lambda}_\alpha$, $\mu = \hat{\mu}_\alpha$, we get

$$\begin{aligned}0 = &\frac{\alpha}{\|\overline{x}_\alpha - \overline{x}\|}[f''_{x\alpha}(\overline{x},0) + g''_{x\alpha}(\overline{x},0)\hat{\lambda}_\alpha + h''_{x\alpha}(\overline{x},0)\hat{\mu}_\alpha]\\ &+H_{(\hat{\lambda}_\alpha,\hat{\mu}_\alpha)}(\overline{x})\Big(\frac{\overline{x}_\alpha - \overline{x}}{\|\overline{x}_\alpha - \overline{x}\|}\Big) + g'_0(\overline{x})\Big(\frac{\lambda_\alpha - \hat{\lambda}_\alpha}{\|\overline{x}_\alpha - \overline{x}\|}\Big) + h'_0(\overline{x})\Big(\frac{\mu_\alpha - \hat{\mu}_\alpha}{\|\overline{x}_\alpha - \overline{x}\|}\Big)\\ &+\frac{o(\|\overline{x}_\alpha - \overline{x}\| + \|(\lambda_\alpha,\mu_\alpha) - (\hat{\lambda}_\alpha,\hat{\mu}_\alpha)\| + \|\alpha\|)}{\|\overline{x}_\alpha - \overline{x}\|}.\end{aligned} \tag{28}$$

Put

$$\nu^1_\alpha = \frac{\lambda_\alpha - \hat{\lambda}_\alpha}{\|\overline{x}_\alpha - \overline{x}\|}, \quad \nu^2_\alpha = \frac{\mu_\alpha - \hat{\mu}_\alpha}{\|\overline{x}_\alpha - \overline{x}\|}, \quad \nu_\alpha = (\nu^1_\alpha, \nu^2_\alpha).$$

According to Theorem 3.1, $\nu_\alpha$ is bounded as $\alpha \to 0$. Consequently, there exists a subsequence $\{\nu_{\alpha_{k_\ell}}\}$ of $\{\nu_{\alpha_k}\}$ such that $\nu_{\alpha_{k_\ell}} \to \overline{\nu} = (\overline{\nu}_1, \overline{\nu}_2) \in \mathbb{R}^m \times \mathbb{R}^s$.

Observing that $\frac{\alpha}{\|\overline{x}_\alpha - \overline{x}\|} \to 0$ as $\alpha \to 0$ and by letting $\ell \to \infty$, it follows from (28) that

$$H_{(\overline{\lambda},\overline{\mu})}(\overline{x})d + g'_0(\overline{x})\overline{\nu}_1 + h'_0(\overline{x})\overline{\nu}_2 = 0,$$

which means that (18) holds.

Moreover, it follows readily from (23) and (24) that

$$\langle \overline{\nu}_1, g'_0(\overline{x})d\rangle = 0, \quad \langle \overline{\nu}_2, h'_0(\overline{x})d\rangle = 0.$$

So we have shown that there exist $0 \neq d \in L$, $(\overline{\lambda}, \overline{\mu}) \in \Lambda(\overline{x})$ and $(\overline{\nu}_1, \overline{\nu}_2) \in \mathbb{R}^m \times \mathbb{R}^s$ such that (18) and (19) are fulfilled. This contradicts the hypotheses. The proof is complete.

**Remark 4.1.** From the proof of Theorem 4.1, we can see that if the assumption of the constraint $g(x,\alpha) \in -S$ of Problem (P$_\alpha$) to be active at $\overline{x}_\alpha$ for $\alpha$ small enough is replaced by $\frac{g(\overline{x}_\alpha,\alpha)}{\|\overline{x}_\alpha - \overline{x}\|} \to 0$ as $\alpha \to 0$, then the conclusion of Theorem 4.1 still holds.

Under the Mangasarian-Fromovitz constraint qualification at $\overline{x}$, Kawasaki [18, Corollary 5.1] have derived a weakly second-order Kuhn-Tucker necessary condition for (P) is as follows:

$\forall\, d \in L,\ \exists\, (\lambda, \mu) \in S^* \times \mathbb{R}^s$ such that

$$f_0'(\overline{x}) + g_0'(\overline{x})\lambda + h_0'(\overline{x})\mu = 0, \tag{29}$$

$$d^T H_{(\lambda,\mu)}(\overline{x})d \geq 0, \tag{30}$$

$$\langle \lambda, g_0(\overline{x})\rangle = 0, \quad \langle \lambda, g_0'(\overline{x})d\rangle = 0, \tag{31}$$

where the superscript $T$ denotes transposition.

In what follows we shall show that when condition (30) becomes strict, i.e.

$$d^T H_{(\lambda,\mu)}(\overline{x})d > 0,$$

then the variation of $x_\alpha$ has the same order of magnitude as the variation of $\alpha$.

**Theorem 4.2.** *Assume that $\overline{x}$ is a local minimizer of Problem* (P)*, and the Mangasarian-Fromovitz condition is fulfilled at $\overline{x}$. Suppose, furthermore, that $f(\cdot,\cdot)$, $g(\cdot,\cdot)$, $h(\cdot,\cdot)$ are functions of the class $C^2$, and the following condition holds:*

$$d^T H_{(\lambda,\mu)} d > 0 \quad (\forall\, 0 \neq d \in L,\ \forall\, (\lambda,\mu) \in \Lambda(\overline{x})). \tag{32}$$

*Then $\frac{\|\overline{x}_\alpha - \overline{x}\|}{\|\alpha\|}$ is bounded as $\alpha \to 0$.*

**Proof.** Contrary to the conclusion, we suppose that $\frac{\|\overline{x}_\alpha - \overline{x}\|}{\|\alpha\|} \to \infty$ as $\alpha \to 0$. We invoke Theorem 4.1 to deduce that there exist $0 \neq d \in L$, $(\overline{\lambda}, \overline{\mu}) \in \Lambda(\overline{x})$ and $(\overline{\nu}_1, \overline{\nu}_2) \in \mathbb{R}^m \times \mathbb{R}^s$ such that

$$H_{(\overline{\lambda},\overline{\mu})}(\overline{x})d + g_0'(\overline{x})\overline{\nu}_1 + h_0'(\overline{x})\overline{\nu}_2 = 0, \tag{33}$$

$$\langle \overline{\nu}_1, g_0'(\overline{x})d\rangle = 0, \quad \langle \overline{\nu}_2, h_0'(\overline{x})d\rangle = 0. \tag{34}$$

It follows readily from (33) that

$$d^T H_{(\overline{\lambda},\overline{\mu})}(\overline{x})d + \langle \overline{\nu}_1, g_0'(\overline{x})d\rangle + \langle \overline{\nu}_2, h_0'(\overline{x})d\rangle = 0. \tag{35}$$

Substituting (34) into (35) yields that

$$d^T H_{(\overline{\lambda},\overline{\mu})}(\overline{x})d = 0,$$

which conflicts with (32).

We now consider the following problem:

$$(P_1) \qquad \begin{cases} \text{minimize } f_0(x), \\ \text{subject to} \\ \ g_{0,i}(x) \le 0, \quad i = 1, \dots, m, \\ \ h_{0,j}(x) = 0, \quad j = 1, \dots, s, \end{cases}$$

where $f_0$, $g_{0,i}$, $h_{0,j}$ $(i = 1, \dots, m; j = 1, \dots, s)$ are real-valued functions defined on $\mathbb{R}^n$. For each parameter $\alpha \in \mathbb{R}^p$, we also consider the perturbed problem for $(\mathrm{P}_1)$:

$$(P_1^\alpha) \qquad \begin{cases} \text{minimize } f(x,\alpha), \\ \text{subject to} \\ \ g_i(x,\alpha) \le 0, \quad i = 1, \dots, m, \\ \ h_j(x,\alpha) = 0, \quad j = 1, \dots, s, \end{cases}$$

where $f(\cdot,\cdot)$, $g_i(\cdot,\cdot)$, $h_j(\cdot,\cdot)$ $(i = 1, \dots, m; j = 1, \dots, s)$ are real-valued functions defined on $\mathbb{R}^n \times \mathbb{R}^p$ such that

$$f(x,0) = f_0(x), \quad g_i(x,0) = g_{0,i}(x), \quad h_j(x,0) = h_{0,j}(x)$$
$$(i = 1, \dots, m;\ j = 1, \dots, s).$$

Let $\overline{x}$ and $\overline{x}_\alpha$ be local minima of Problem $(\mathrm{P}_1)$ and $(\mathrm{P}_1^\alpha)$, respectively, and $\overline{x}_\alpha \to \overline{x}$. Suppose that the constraints $g_{0,i}(x)$ and $g_i(x,\alpha)$ $(i = 1, \dots, m)$ are active at $\overline{x}$ and $\overline{x}_\alpha$ for $\alpha$ small enough, respectively. We set

$$H^1_{(\lambda,\mu)}(\overline{x}) = f_0''(\overline{x}) + \sum_{i=1}^m \lambda_i g_{0,i}''(\overline{x}) + \sum_{j=1}^s \mu_j h_{0,j}''(\overline{x}),$$
$$L_1 = \{d \in \mathbb{R}^n : f_0'(\overline{x})d = 0,\ g_{0,i}'(\overline{x})d \le 0,\ h_{0,j}'(\overline{x})d = 0,$$
$$i = 1, \dots, m;\ j = 1, \dots, s\},$$
$$C = \{d \in \mathbb{R}^n : f_0'(\overline{x})d \le 0,\ g_{0,i}'(\overline{x})d \le 0,\ h_{0,j}'(\overline{x})d = 0,$$
$$i = 1, \dots, m;\ j = 1, \dots, s\},$$

where $\lambda = (\lambda_1, \dots, \lambda_m)$, $\mu = (\mu_1, \dots, \mu_s)$.

Assume that $f_0(\cdot)$, $g_{0,i}(\cdot)$, $h_{0,j}(\cdot)$, $f(\cdot,\cdot)$, $g_i(\cdot,\cdot)$, $h_j(\cdot,\cdot)$ $(i = 1, \dots, m; j = 1, \dots, s)$ are $C^2$-functions.

From Theorem 4.2 we obtain a result due to Bonnans [4] as a special case.

**Corollary 4.1.**([4]) *Assume that* $\overline{x}$ *is a local minimizer of Problem* $(P_1)$ *which satisfied the Mangasarian-Fromovitz constraint qualification. Suppose also that*

$$d^T H_{(\lambda,\mu)}(\overline{x})d > 0 \quad (\forall\, 0 \neq d \in C,\ \forall\, (\lambda, \mu) \in \Lambda_1(\overline{x})), \tag{36}$$

*where* $\Lambda_1(\overline{x})$ *denotes the set of Lagrange multipliers at* $\overline{x}$ *of* $(P_1)$*. Then* $\frac{\|\overline{x}_\alpha - \overline{x}\|}{\|\alpha\|}$ *is bounded as* $\alpha \to 0$*.*

**Proof.** Since condition (36) implies the following condition:

$$d^T H_{(\lambda,\mu)}(\overline{x})d > 0 \quad (\forall\, 0 \neq d \in L_1, \forall\, (\lambda, \mu) \in \Lambda_1(\overline{x})),$$

applying Theorem 4.2, we deduce the desired conclusion.

## References

[1] A. Auslender, *Differentiable stability in nonconvex and nondifferentiable programming*, Math. Programming Study, 10 (1979), 29-41.

[2] A. Ben-Tal and J. Zowe, *A unified theory of first and second order conditions for extremum problems in topological vector spaces*, Math. Programming Study, 19 (1982), 39-76.

[3] J. F. Bonnans, *A semi-strong sufficiency condition for optimality in nonconvex programming and its connection to the perturbation problem*, J. Optim. Theory Appl., 60(1989), 7-18.

[4] J. F. Bonnans, *On the stability of solutions in nonlinear programming*, Optimization, 21:3 (1990), 365-370.

[5] J. F. Bonnans, *Directional derivatives of optimal solutions in smooth nonlinear programming*, J. Optim. Theory Appl., 73:1 (1992), 27-45.

[6] B. Cornet and G. Laroque, *Lipschitz properties of solutions in mathematical programming*, J. Optim. Theory Appl., 53 (1987), 407-411.

[7] B. D. Craven, *Nondifferentiable optimization by smooth approximations*, Optimization, 17(1986), 3-17.

[8] B. D. Craven, *Convergence of discrete approximations for constrained minimization*, J. Austral. Math. Soc., Series B, 25 (1994), 1-12.

[9] B. D. Craven and D. V. Luu, *A method for establishing optimality conditions for a nonsmooth vector-valued minimax problem*, J. Optim. Theory Appl., 95 (1997), 295-304.

[10] B. D. Craven and D. V. Luu, *Lagrangian conditions for a nonsmooth vector-valued minimax*, J. Austral. Math. Soc., Series A, 65 (1998), 163-175.

[11] B. D. Craven and D. V. Luu, *Perturbing convex multiobjective programs*, Optimization, 48(2000), 391-407.

[12] A. V. Fiacco, *Sensitivity analysis for nonlinear programming using penalty methods*, Math. Programming, 10 (1976), 287-311.

[13] J. Gauvin and R. Janin, *Directional behavior of optimal solutions in nonlinear mathematical programming*, Mathematics of Operators Research, 13 (1988), 629-649.

[14] I. V. Girsanov, Lectures on mathematical theory of extremum problems, Lecture Notes in Economics and Mathematical Systems 67, Springer-Verlag, New York, 1972.

[15] D. Hoa and D. V. Luu, *On the stability of local minima in nonsmooth mathematical programs*, East-West J. of Mathematics, 4:1 (2002), 1-13.

[16] A. D. Ioffe and V. M. Tikhomirov, Theory of Extremal Problems, Izdat. Nauka, Moscow, 1974 (Russian).

[17] J. Jittorntrum, *Solution point differentiability without strict complementarity in nonlinear programming*, Math. Programming Study, 21 (1984), 127-138.

[18] H. Kawasaki, *An envelope-like effect of infinitely many inequality constraints on second-order necessary conditions for minimization problems*, Math. Programming, 41 (1988), 73-96.

[19] O. L. Mangasarian and S. Fromovitz, *The Fritz John necessary optimality conditions in the presence of equality and inequality constraints*, J. Math. Anal. Appl., 7 (1967), 37-47.

[20] S. M. Robinson, *Stability theory for systems of inequalities, Part II: Differentiable nonlinear systems*, SIAM J. Numer. Anal., 13 (1976), 497-513.

[21] S. M. Robinson, *Generalized equations and their solutions, Part II: Applications to nonlinear programming*, Math. Programming Study, 19 (1982), 200-221.

[22] R. T. Rockafellar, Convex Analysis, Princeton University Press, Princeton, New Jersey, 1970.

[23] M. Studniarski, *Sufficient conditions for the stability of local minimum points in nonsmooth optimization*, Optimization, 20 (1989), 27-35.

[24] J. Zowe and S. Kurcyusz, *Regularity and stability of the mathematical programming problem in Banach spaces*, Appl. Math. Optim., 5 (1979), 49-62.

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