Abstract. The spaces of entire functions represented by Dirichlet series have been studied earlier by many workers. The aim of this paper is to study the bornological properties of the space $\Gamma$ of entire functions represented by Dirichlet series. By $\Gamma$ we denote the space of all analytic functions $\alpha(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$, having finite abscissa of convergence. We introduce bornologies on $\Gamma$ and $\overline{\Gamma}$, and prove that $\overline{\Gamma}$ is a convex bornological vector space which is the completion of the convex bornological vector space $\Gamma$.

1. Let $X$ be the space of entire functions $\sum a_n z^n$ defined over the complex field $\mathbb{C}$.

Defining the norm $\|\alpha\| = \sup\{|a_0|, |a_n|^{1/n}, n \geq 1\}$ on $X$, Patwardhan [4] introduced the concept of bornology on this space. In this paper, we consider the space of entire function $\alpha(s)$ represented by Dirichlet series. We begin by considering the Dirichlet series

$$\alpha(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n),$$

(1.1)

where $\lambda_{n+1} > \lambda_n$, $\lambda_1 \geq 0$, $\lim_{n \to \infty} \lambda_n = \infty$, $s = \sigma + it$, ($\sigma$, $t$ real variables), and \{a_n\} is any sequence of complex numbers. Set

$$\lim_{n \to \infty} \sup \frac{\log n}{\lambda_n} = D^*.$$

(1.2)
It is well known that if \( D^* < \infty \) and
\[
c = \liminf_{n \to \infty} \frac{\log |a_n|^{-1}}{\lambda_n} = \infty
\] (1.3)
then \( \alpha(s) \) represents an entire function and the series (1.1) converges absolutely at every point of the finite complex plane. We now assume that the exponents \( \lambda_n \)'s satisfy the relation
\[
\limsup_{n \to \infty} \frac{n}{\lambda_n} = D < \infty.
\] (1.4)

For an entire function \( \alpha(s) \), we define the number \( \| \alpha \| \) by
\[
\| \alpha \| = l.u.b.\{|a_n|^{1/\lambda_n}\}, \quad n \geq 1.
\] (1.5)

It is easily verified that \( \| \alpha \| \) satisfies the following conditions:

(i) \( \| \alpha \| \geq 0 \) and \( \| \alpha \| = 0 \) if and only if \( \alpha = \theta \), the identically zero function;
(ii) \( \| \alpha + \beta \| \leq \| \alpha \| + \| \beta \| \);
(iii) \( \| w\alpha \| \leq A(w)\| \alpha \| \), where \( A(w) = \max(1, |w|) \), \( w \) being any complex number.

(1.6)

It follows from (i) and (ii) of (1.6) that \( \rho(\alpha, \beta) = \| \alpha - \beta \| \) defines a metric in the class of entire functions.

2. In this section we give some definitions. We have

2.1. A bornology on a set \( X \) is a family \( \mathcal{B} \) of subsets of \( X \) satisfying the following axioms:

(i) \( \mathcal{B} \) is a covering of \( X \), i.e. \( X = \bigcup_{B \in \mathcal{B}} B \);
(ii) \( \mathcal{B} \) is hereditary under inclusion, i.e. if \( A \in \mathcal{B} \) and \( B \) is a subset of \( X \) contained in \( A \), then \( B \in \mathcal{B} \);
(iii) \( \mathcal{B} \) is stable under finite union.

A pair \((X, \mathcal{B})\) consisting of a set \( X \) and a bornology \( \mathcal{B} \) on \( X \) is called a bornological space, and the elements of \( \mathcal{B} \) are called the bounded subsets of \( X \).

2.2. Let \( E \) be a bornological vector space. A sequence \( \{x_n\} \) in \( E \) is said to be Mackey convergent to a point \( x \in E \) if there exists a decreasing sequence \( \{t_n\} \) of
positive real numbers tending to zero such that the sequence \( \{ \frac{x_n - x}{t_n} \} \) is bounded. In the sequel, we shall refer Mackey convergence as \( M \)-convergence.

2.3. Let \( E \) be a separated convex bornological space. A sequence \( \{x_n\} \) in \( E \) is said to be a bornological Cauchy sequence (or a Mackey-Cauchy sequence) in \( E \) if there exists a bounded disk \( B \subset E \) such that \( \{x_n\} \) is a Cauchy sequence in \( E_B \).

For further definitions we shall refer to [2], [3] and [5].

3. The Space \( \Gamma \)

We shall denote by \( \Gamma \) the space of all entire functions \( \alpha(s) \) having representation \((1.1)\) and satisfying the conditions \((1.3)\) and \((1.4)\) and topologized by the metric \( \rho \) mentioned in \((1.6)\). We define a bornology on \( \Gamma \) with the help of \( \| \cdot \| \) defined in \((1.5)\). We denote by \( B_k \) the set \( \{ \alpha \in \Gamma : \| \alpha \| \leq k \} \). Then the family \( B_0 = \{ B_k : k = 1, 2, \ldots \} \) forms a base for a bornology \( B \) on \( \Gamma \). We now have:

**Theorem 1.** \((\Gamma, B)\) is a separated convex bornological vector space with a countable base.

The proof runs on the same lines as that of Theorem 3.1 of [5]. Hence we omit the proof.

Next we have:

**Theorem 2.** \( B \) contains no bornivorous set.

The proof follows as in [5, Theorem 3.2].

The following result is due to H. Hogbe and Nlend [2].

**Theorem 3.** The \( M \)-convergence in a bornological vector space \( E \) is topologizable if and only if \( E \) has a bounded bornivorous set.

Combining Theorems 2 and 3, we get the following:

**Corollary 1.** The \( M \)-convergence of \( \Gamma \) is not topologizable.

4. In this section we consider the properties of linear functionals defined on the
space \( \Gamma \). We prove:

**Theorem 4.** Every continuous linear functional \( f \) defined on \( \Gamma \) is of the form

\[
f(\alpha) = \sum_{1}^{\infty} c_n a_n, \text{ where } \alpha = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}, \text{ and } \{ |c_n|^{1/\lambda_n} \} \text{ is a bounded sequence.}
\]

To prove this we need the following:

**Lemma 1.** A necessary and sufficient condition that \( \sum_{n=1}^{\infty} c_n a_n \) should be convergent for every sequence \( \{ a_n \} \) satisfying

\[
|a_n|^{1/\lambda_n} \to 0 \quad \text{as } n \to \infty,
\]

is that the sequence \( \{ |c_n|^{1/\lambda_n} \}_{n=1}^{\infty} \) be bounded.

**Proof.** Suppose that \( \{ |c_n|^{1/\lambda_n} \}_{n=1}^{\infty} \) is bounded. Then we can find \( M > 0 \) such that \( |c_n|^{1/\lambda_n} \leq M \) for \( n \geq 1 \). By (4.1), we can find \( n_0 \) such that \( |a_n|^{1/\lambda_n} \leq \frac{1}{2M} \), \( n \geq n_0 \).

Hence \( |c_n a_n| \leq \frac{1}{2M} \), \( n \geq n_0 \). Thus we have

\[
\left| \sum_{n_0+1}^{\infty} a_n c_n \right| \leq \sum_{n_0+1}^{\infty} |a_n c_n| < \sum_{n_0+1}^{\infty} 2^{-\lambda_n} \leq \sum_{1}^{\infty} 2^{-\lambda_n}.
\]

In view of (1.4), \( -\lambda_n < -\frac{\pi}{2(D+c)} \), \( n \geq n_0 \); hence \( \sum 2^{-\lambda_n} \leq \sum 2^{-n/(D+c)} < \infty \). Therefore \( \sum_{n=1}^{\infty} c_n a_n \) converges.

To prove the converse, suppose that the sequence \( \{ |c_n|^{1/\lambda_n} \} \) is unbounded and let \( p > 0 \) be sufficiently large. Then we can find an increasing sequence of integers \( \{ n_p \} \) such that \( |c_{n_p}| \geq p^{\lambda_{n_p}} \), \( p = 1, 2, \ldots \). We define the sequence \( \{ a_n \} \) as

\[
a_n = 0 \text{ if } n \neq n_p \text{ and } a_n = 1/p^{\lambda_n} \text{ if } n = n_p, \quad p = 1, 2, \ldots.
\]

Then \( \{ a_n \} \) satisfies (4.1) but \( |c_n a_n| \geq 1 \) for \( n = n_p \), so that \( \sum_{n=1}^{\infty} c_n a_n \) does not converge.

This proves Lemma 1.
Proof of Theorem 4. Let \( \alpha = \sum_{n=1}^{\infty} a_n e^{\lambda_n} \) and \( f \) be a continuous linear functional on \( \Gamma \). Let \( \alpha_n = e^{\lambda_n} \) and \( f(\alpha_n) = c_n \). Then

\[
f(\alpha) = \lim_{n \to \infty} f(a_1 \alpha_1 + a_2 \alpha_2 + \cdots + a_n \alpha_n) = \lim_{n \to \infty} (c_1 a_1 + c_2 a_2 + \cdots + c_n a_n).
\]

Thus for every \( \alpha \in \Gamma \), \( \sum_{n=1}^{\infty} c_n a_n \) converges and \( f(\alpha) = \sum_{n=1}^{\infty} c_n a_n \). Hence by Lemma 1, \( \{ |c_n|^{1/\lambda_n} \} \) is bounded.

Conversely, suppose that \( \{c_n\} \) is a sequence of complex numbers such that \( \{ |c_n|^{1/\lambda_n} \} \) is bounded. For any entire function \( \alpha = \sum_{n=1}^{\infty} a_n e^{\lambda_n}, |a_n|^{1/\lambda_n} \to 0 \) as \( n \to \infty \) and hence by Lemma 2, the series \( \sum_{n=1}^{\infty} a_n c_n \) is convergent. For entire functions \( \alpha \in \Gamma \), we define the functional \( f \) by \( f(\alpha) = \sum_{n=1}^{\infty} c_n a_n \). Then \( f \) is obviously linear on \( \Gamma \). We shall show that it is continuous. For this purpose it is enough to show that, if for any sequence \( |a_p| \to 0 \) as \( p \to \infty \), then \( f(a_p) \to 0 \). For this purpose, let

\[
\alpha_p = \sum_{n=1}^{\infty} a_{p_n} e^{\lambda_n}, \quad p = 1, 2, \ldots
\]

Since \( \{ |c_n|^{1/\lambda_n} \} \) is bounded, we can find \( M \) such that \( |c_n| \leq M^{\lambda_n}; n \geq 1 \). Given \( \varepsilon > 0 \), choose \( \eta \) so that \( 0 < \eta < \frac{1}{M} \) and \( \eta M(1 + \frac{1}{1-\eta M}) < \varepsilon \). Since \( |a_p| \to 0 \), we can find \( p_0 \) so that \( |a_p| \leq \eta \) for \( p \geq p_0 \). Hence

\[
|f(\alpha_p)| \leq \eta M + \sum_{n=1}^{\infty} (\eta M)^{\lambda_n} = \eta M \left( 1 + \frac{1}{1-\eta M} \right) < \varepsilon \quad \text{for } p \geq p_0.
\]

Hence \( f(\alpha_p) \to 0 \) as \( p \to \infty \). This completes the proof of Theorem 4.

The following result was given by Hogbe-Nlend [2]:

Lemma 2. A linear functional \( f : \Gamma \to C \) is bounded if and only if \( f \) maps every \( M \)-convergent sequence to a bounded sequence in \( C \).

Now we have:

Theorem 5. A linear functional \( f \) on \( \Gamma \) defined by \( f(\alpha) = \sum_{n=1}^{\infty} a_n c_n \) is bounded if and only if \( \lim_{n \to \infty} |c_n|^{1/\lambda_n} = 0 \).

Proof. Suppose \( |c_n|^{1/\lambda_n} \to 0 \). Let \( \{\alpha_q\} \) be a sequence in \( \Gamma \) such that \( \alpha_q \xrightarrow{M} 0 \). Then there exists a positive constant \( y \) and a decreasing sequence \( \{t_q\} \) of scalars.
converging to zero such that \(|\alpha_q/t_q| \leq y\), i.e. \(|a_{q,n}| \leq |t_q|y^\lambda_n, n \geq 1\). Since \(|c_n|^{1/\lambda_n} \to 0\), there exists \(n_0\) such that \(|c_n|^{1/\lambda_n} \leq \frac{1}{y^{\lambda}}\) for all \(n \geq n_0\). Hence \(|c_n| \leq (2y)^{\lambda_n}, n \geq n_0\).

Now
\[
|f(\alpha_q)| = \left| \sum_{n=1}^{\infty} c_n a_{q,n} \right| \leq \sum_{n=1}^{\infty} |c_n| |a_{q,n}| \leq \sum_{n=1}^{\infty} |c_n| |t_q|y^{\lambda_n} \\
\leq \sum_{n=1}^{n_0-1} |c_n| |t_q|y^{\lambda_n} + \sum_{n=n_0}^{\infty} |t_q|2^{-\lambda_n} < \infty.
\]

Thus the sequence \(\{f(\alpha_q)\}\) is bounded. Hence \(f\) is bounded on every sequence which \(M\)-converges to zero and consequently by Lemma 2, \(f\) is bounded.

Conversely, let \(f\) be such that \(\limsup_{n \to \infty} |c_n|^{1/\lambda_n} = \rho > 0\). Then given \(\eta > 0\) such that \(\eta < \rho\), there exists a divergent increasing sequence \(\{n_q\}\) of integers such that \(|c_n|^{1/\lambda_n} > \eta\) for all \(n = n_q\). Choose \(\psi \in R\) such that \(\psi > 1\) and \(\psi\eta > 1\). Consider the sequence \(\{\alpha_n\}\) where \(\alpha_n = \psi^n e^{\lambda_n} \in \Gamma\) and define \(t_n \in C\) as \(t_n = \frac{1}{\psi^{\lambda_n}}\). Then \(t_n \to 0\) as \(n \to \infty\) and \(|\alpha_n/t_n| = \|\psi^{2\lambda_n} e^{\lambda_n}\| = |\psi|^2 < \infty\). Consequently \(\alpha_n \to 0\). But \(f(\alpha_n) = c_n \psi^{\lambda_n}\) and \(|f(\alpha_{n_q})| = |c_{n_q}| |\psi|^\lambda_{n_q} > \eta^{\lambda_{n_q}} |\psi|^{\lambda_{n_q}}\) which is not bounded. Hence by Lemma 2, \(f\) is not bounded. This proves sufficiency part and proof of Theorem 5 is complete.

5. \(\sigma\)-Norms on \(\Gamma\)

We define, for each \(\sigma < \infty\), the expression \(\|\alpha : \sigma\|\) by equation
\begin{equation}
\|\alpha : \sigma\| = \sum_{n=1}^{\infty} |a_n| e^{\sigma \lambda_n}.
\end{equation}

It is easily seen that, for each \(\sigma\), (5.1) defines a norm on the class of entire functions represented by Dirichlet series. We shall denote by \(\Gamma(\sigma)\) the space \(\Gamma\) endowed by this norm. We denote by \(B_\sigma\) the bornology on \(\Gamma\) consisting of the sets bounded in the sense of the norm \(\|\alpha : \sigma\|\). We now prove:

**Theorem 6.** \(B = \bigcup_{\sigma < \infty} B_\sigma\).

**Proof.** Let \(G \in B\). Then there exists a constant \(J\) such that \(\|\alpha\| \leq J\) for all \(\alpha \in G\). Let \(\alpha = \sum_{n=1}^{\infty} a_n e^{\lambda_n} \in G\). Then \(|a_n| e^{\lambda_n} \leq J\lambda_n e^{\lambda_n}, n \geq 1\). Thus if
$|e^s| < \frac{1}{J}$, we have,

$$
\sum_{n=1}^{\infty} |a_n| |e^{s\lambda_n}| \leq \sum_{n=1}^{\infty} J^{\lambda_n} |e^s|^{\lambda_n} < \infty.
$$

Hence if $0 < e^\sigma < \frac{1}{J}$ then $B \in B_\sigma$ and so $B \subset \cup_{\sigma<\infty} B_\sigma$.

For the reverse inclusion let $G \in B_\sigma$, then there exists a constant $J$ such that for all $G$, $\|\alpha \cdot \sigma\| \leq J$,

i.e. $\sum_{n=1}^{\infty} |a_n| e^{\sigma \lambda_n} \leq J$ i.e. $|a_n|^{1/\lambda_n} \leq J^{1/\lambda_n} \cdot e^{-\sigma}$, $n \geq 1$,

i.e. $\|\alpha\| \leq \text{l.u.b.}\{J^{1/\lambda_n} \cdot e^{-\sigma}, \ n \geq 1\} < \infty$.

Thus $G \in B$ and hence $\cup_{\sigma<\infty} B_\sigma \subset B$. This completes the proof of Theorem 6.

Now we prove

**Lemma 3.** Let $\{\alpha_n\}$ be a sequence of entire functions in $(\Gamma, B)$. The following statements are equivalent for $(\Gamma, B)$:

(i) $\alpha_n \xrightarrow{M} 0$.

(ii) There exists a sequence $\{t_n\}$ of positive real numbers tending to zero such that $\{\alpha_n/t_n\}$ is bounded.

**Proof.** (i)$\rightarrow$(ii) is obviously true. To prove (ii)$\rightarrow$(i), let $\{\alpha_n\}$ be a sequence in $\Gamma$ for which there exists a sequence $\{t_n\}$ of positive real numbers tending to zero and a constant $J$ such that $\|\alpha_n/t_n\| \leq J$ for all $n$. Now there exists a positive number $M$ such that $t_n \leq M$ for all $n$. Further, we can choose for each $i = 1, 2, \ldots$ an integer $n_i$ such that $t_n < 1/i$ for all $n \geq n_i$. Let us define a sequence $\{t'_n\}$ as

$$
t'_n = \begin{cases} M \quad \text{for all } n < n_1, \\ 1/i \quad \text{for all } n \leq n_i < n_{i+1}, \quad i = 1, 2, \ldots.
\end{cases}
$$

Then $\{t'_n\}$ is a decreasing sequence of positive real numbers tending to zero and further $t'_n \geq t_n$ for all $n$. Hence $\|\alpha_n/t'_n\| = \|\alpha_n t_n/t'_n t'_n\| < A(t_n/t'_n)\|\alpha_n/t_n\| \leq J$. Therefore $\alpha_n \xrightarrow{M} 0$. Hence (ii)$\rightarrow$(i) and proof of Lemma 3 is complete.

Let $\{\alpha_q\}$ be a sequence of entire functions in $\Gamma$. Then we have:
Theorem 7. \( \alpha_q \xrightarrow{M} 0 \) in \( \Gamma \) if and only if \( \alpha_q(s) \to 0 \) uniformly in some finite half plane.

Proof. Suppose \( \alpha_q \xrightarrow{M} 0 \) and \( \alpha_q = \sum_{n=1}^{\infty} a_{q,n} e^{\lambda_n} \). Then there exists a constant \( J \) and a sequence \( \{t_q\} \) in \( C \), tending to zero such that \( \|a_{q,n}\| \leq J \) for all \( q \).

i.e. \( \left| \frac{a_{q,n}}{t_q} \right| \leq J^{\lambda_n}, \ n \geq 1 \). If \( s \in C \) such that \( |e^s| \leq 1/2J \), then

\[
\|\alpha_q(e^s)\| = \left\| \sum_{n} a_{q,n} e^{\lambda_n} \right\| \leq \sum_{n} \|a_{q,n}\| \|e^s\|^{\lambda_n} \leq \sum_{n} \|t_q\| J^{\lambda_n} \|e^s\|^{\lambda_n} \leq |t_q|.
\]

Hence \( \|\alpha_q(s)\| \to 0 \) uniformly for all \( s \) such that \( |e^s| \leq \frac{1}{2J} \).

Conversely, suppose that there exists \( \sigma_0 < \tau \) such that \( \alpha_q(s) \to 0 \) uniformly for all \( s = \sigma + it \) satisfying \( \sigma < \sigma_0 \). Then

\( l.u.b|\alpha_q(e^s)| \to 0 \) as \( q \to \infty \).

Now \( |\alpha_q(s)| \leq l.u.b|\alpha_q(s)| \) for all \( s \) such that \( \sigma < \sigma_0 \). Hence \( |a_{q,n}| e^{\sigma \lambda_n} \leq l.u.b|\alpha_q(s)| \)

i.e.

\[
\left[ \frac{|a_{q,n}|}{l.u.b|\alpha_q(s)|} \right]^{1/\lambda_n} \leq e^{-\sigma}.
\]

Let \( t_q = l.u.b|\alpha_q(s)| \). Then \( |a_{q,n}| = |\alpha_q(n)| \leq t_q \) and \( t_q \to 0 \). Hence

\[
\left\| \frac{\alpha_q}{t_q} \right\| = l.u.b \left\{ \frac{|a_{q,n}|^{1/\lambda_n}}{t_q} \right\} \leq \max\{1, e^{-\sigma}\} = A(e^{-\sigma}),
\]

and hence \( \alpha_q \xrightarrow{M} 0 \) in view of Lemma 4. This proves Theorem 7.

We now obtain some properties of linear functions on \( \Gamma \). We prove

Lemma 4. In the topological dual \( \Gamma' \) of \( \Gamma \), every continuous linear functional \( f \) is of the form \( f(\alpha) = \sum_{n=1}^{\infty} c_n a_n, \ \alpha = \sum_{n=1}^{\infty} a_n e^{\lambda_n} \), if and only if the sequence \( \{|c_n| e^{-\sigma \lambda_n}\} \) is bounded.

Proof. Suppose that \( f(\alpha) \) is continuous linear functional on \( \Gamma \). Then there exists \( k > 0 \) such that \( |f(\alpha)| \leq k\|\alpha : \sigma\| \) for every \( \alpha \). Let \( \delta_n = e^{\sigma \lambda_n} \) and \( f(\delta_n) = c_n \).
In $\Gamma$, $\alpha = \sum_{n=1}^{\infty} a_n e^{s \lambda_n} = \lim_{n \to \infty} \sum_{i=1}^{n} a_i \delta_i$. Since $f$ is continuous, we have

$$f(\alpha) = f \left( \lim_{n \to \infty} \sum_{i=1}^{n} a_i e^{s \lambda_i} \right) = \lim_{n \to \infty} \sum_{i=1}^{n} a_i f(\delta_i) = \sum_{i=1}^{\infty} a_i c_n.$$ 

Also $|c_n| \leq k \|\delta_n : \sigma\| = ke^{\sigma \lambda_n}$. Hence $\{|c_n|e^{-\sigma \lambda_n}\}$ is bounded.

Conversely let $\{c_n|e^{-\sigma \lambda_n}\}_{n \geq 1}$ be bounded. Let $f$ be defined by $f(\alpha) = \sum_{n=1}^{\infty} c_n a_n$, $\alpha = \sum_{n=1}^{\infty} a_n e^{\lambda_n s}$, then $f$ is a linear functional and

$$|f(\alpha)| \leq \sum_{n=1}^{\infty} |c_n| |a_n| < \sum_{n=1}^{\infty} k e^{\sigma \lambda_n} |a_n| \quad \text{(for some } k > 0) = k \|\alpha : \sigma\|. $$

Hence $f(\alpha)$ is continuous on $\Gamma$. This proves Lemma 4.

**Theorem 8.** The bornological dual $\Gamma^*$ of $\Gamma$ is the same as its topological dual $\Gamma'$.

**Proof.** The proof follows immediately from the fact that a linear functional on a normed linear space is continuous if and only if it is bounded.

On $\Gamma'$ we now define a map:

$$\| : \frac{1}{2\sigma} \| : \Gamma' \to \mathbb{R}, \quad \alpha = \sum_{n=1}^{\infty} a_n e^{s \lambda_n} \to \| \alpha : \frac{1}{2\sigma} \| = \sum_{i=1}^{\infty} |a_n| (2e^{\sigma})^{-\lambda_n}.$$  

By Lemma 4 it follows that $\alpha = \sum_{n=1}^{\infty} a_n e^{s \lambda_n} \in \Gamma'$ if and only if $\{|a_n|e^{-\sigma \lambda_n}\}$ is bounded. Consequently the function $\| : \frac{1}{2\sigma} \|$ is well defined and $\Gamma'$ becomes a normed linear space relative to $\| : \frac{1}{2\sigma} \|$. We denote by $\overline{B}_{1/2\sigma}$ the canonical bornology of $\Gamma'$ with this norm, which we call the $(\frac{1}{2\sigma})$-norm.

6. In this section we consider the set 

$$\Gamma = \left\{ \beta = \sum_{n=1}^{\infty} b_n e^{s \lambda_n} : b_n \in C \text{ and } \{|b_n|^{1/\lambda_n}\} \text{ is bounded} \right\}.$$ 

A convex bornology $\overline{B}$ can be defined on $\Gamma$ with the help of a function $\| : \| : \Gamma \to \mathbb{R}$ defined in a similar fashion to that on $\Gamma$. We note that $\overline{B}$ when restricted to $\Gamma$ gives $B$. Moreover, $\Gamma = \bigcup_{\sigma < \infty} \Gamma(\sigma)$, and as in the proof of Theorem 7, we have $\overline{B} = \bigcup_{\sigma < r} \overline{B}(\sigma)$.

We now prove:
Theorem 9. \((\mathcal{F}, \mathcal{B})\) is \(M\)-complete.

Proof. We first observe that Lemma 3 holds for \(\mathcal{F}\) also. Let thus \(\{\alpha_n\}\) be an \(M\)-Cauchy sequence in \(\mathcal{F}\). Then there exists a sequence \(\{\mu_{nm}\}\) of scalars, tending to zero, such that \(\|\frac{\alpha_n - \alpha_m}{\mu_{nm}}\| \leq J\), where \(J\) is some fixed real positive number.

Now we choose a sequence \(\{t_{nm}\}\) of scalars, such that \(t_{nm} \geq \mu_{nm}\) for all \(n, m\) and for \(n_1 \geq n_2, m_1 \geq m_2, t_{n_1m_1} < t_{n_2m_2}\). Since \(\mu_{nm} \to 0\), without loss of generality we can assume that \(\mu_{nm} < 1\) for all \(n, m\). Now we set \(n_1 = 1, m_1 = 1\) and choose \((n_i, m_i)\) inductively such that \(n_i > n_{i-1}, m_i > m_{i-1}\) and \(\mu_{nm} < 1/i\) for \(n \geq n_i, m \geq m_i\). Define the double sequence \(\{t_{nm}\}\) as

\[
t_{nm} = \frac{1}{\min(i, j)} \text{ if } n_i \leq n < n_{i+1} \text{ and } m_j \leq m < m_{j+1}.
\]

It is easily seen that \(\{t_{nm}\}\) is the required sequence. Moreover, \(t_{nm} \to 0\) and

\[
\|\frac{\alpha_n - \alpha_m}{t_{nm}}\| \leq \|\frac{\alpha_n - \alpha_m}{\mu_{nm}}\| \leq J
\]

i.e. \(\frac{|a_{n,q} - a_{n,q}|}{t_{nm}} \leq J\) for all \(q \geq 1\)

i.e. \(\{a_{n,q}\}, q \geq 1\), are Cauchy sequences and hence there exist,

\(a_q, q \geq 1\) in \(C\) such that \(a_{n,q} \to a_q\) for all \(q \geq 1\).

Now \(\frac{|a_{n,q} - a_{p,q}|}{|t_{n,p}|} \leq \frac{|a_{n,q} - a_{p,q}|}{|t_{n,n+1}|} \leq J\) for all \(p \geq n + 1\).

Hence as \(p \to \infty\), we get \(\frac{|a_{n,1} - a_1|}{t_{n,n+1}} \leq J\). Similarly \(\frac{|a_{n,q} - a_q|/q}{t_{n,n+1}} \leq J\) for all \(q \geq 1\)

i.e. \(\frac{\|\alpha_n - \alpha\|}{t_{n,n+1}} \leq J\) where \(\alpha = \sum a_n e^{s\lambda_n}\) and \(t_{n,n+1} \to 0\). Hence \(\alpha \frac{M}{\mu_{n+1}}\).

Now \(\|a_n\|^{1/\lambda_n} = |a_{n,q} - a_n|^{1/\lambda_n} < |a_{n,q} - a_n|^{1/\lambda_n} + |a_{n,q}|^{1/\lambda_n}\)

\[
\leq |t_{q,q+1}|J + |a_{n,q}|^{1/\lambda_n}.
\]

Hence \(\limsup_{n \to \infty} |a_n|^{1/\lambda_n} \leq J \limsup_{n \to \infty} |t_{q,q+1}| + \limsup_{n \to \infty} |a_{n,q}|^{1/\lambda_n} \leq MJ + \|\alpha_q\| < \infty,
\]

where \(M = l.u.b. |t_{q,q+1}| < \infty\). Hence \(\alpha \in \mathcal{F}\) and therefore \(\mathcal{F}\) is \(M\)-complete.

Corollary 2. \(\mathcal{F}\) is complete.
Proof. In view of Theorem 1 in [2, p.33] it is enough to show that $B$ is $l^1$-discd. For this we show that each $B_\sigma \in B$ is $l^1$-discd. Let thus $\{t_\i\}$ be a sequence of scalars such that $\sum_{\i=1}^\i |t_\i| \leq 1$, and $\{\alpha_\i\}$ be a sequence in $B_\sigma$.

Then $\|\alpha = \sum_1^\i t_\i \alpha_\i\| = \text{l.u.b.}\left\{\sum_1^\i |t_\i| |a_\i\| \cdot \lambda_\i, \; \i \geq 1\right\} \leq \text{l.u.b.}\left\{\sum_1^\i |t_\i| \cdot \lambda_\i, \; \i \geq 1\right\} \leq J.$

Hence $B_\sigma$ is $l^1$-discd and Corollary 2 follows.

Theorem 10. $(\Gamma, B)$ is not complete.

Proof. In view of a result by H. Hogbe-Nlend [2], it is enough to show that $(\Gamma, B)$ is not $M$-complete. Consider the sequence, $\alpha_\i = \sum_{k=1}^\i 2^{-k} e^{k \lambda_\i}, \; \i \geq 1$. Then $\{\frac{n-\i}{(1/2)^m}, \; \i \geq m\}$ is bounded in $\Gamma$. In other words, $\{\alpha_\i\}$ is an $M$-Cauchy sequence in $\Gamma$ and hence in $\Gamma$. As $(\Gamma, B)$ is $M$-complete, the $M$-limit of $\{\alpha_\i\}$ exists in $\Gamma$. In fact the $M$-limit of $\{\alpha_\i\}$ in $\Gamma$ is $\alpha = \sum_{k=1}^\i (1/2)^k (e^{k \lambda_\i})$ as $\{(\alpha_\i - \alpha)/2^\i\}$ is bounded in $\Gamma$, and $\alpha \in \Gamma$.

We now claim that the $M$-limit of the sequence $\{\alpha_\i\}$ does not exist in $\Gamma$. For otherwise, let $\alpha_\i \xrightarrow{M} \beta \in \Gamma$. Then $\alpha_\i \xrightarrow{M} \beta$ in $\Gamma$. Hence $\beta = \alpha$ as $\Gamma$ is a separated bornological vector space. This contradicts the fact that $\alpha \notin \Gamma$. Hence $(\Gamma, B)$ is not $M$-complete.

Lastly we have:

Theorem 11. $(\Gamma, B)$ is the $M$-completion of $(\Gamma, B)$.

Proof. Let $\alpha = \sum_{n=1}^\i c_\i e^{\lambda_\i} \in \Gamma$. Then there exists a number $h$ such that $|c_\i|^{1/\lambda_\i} < h$ for all $\i \geq 1$. Now we consider the sequence $\{\alpha_\i = \sum_{n=1}^\i c_\i e^{\lambda_\i}, \; q = 1, 2, \ldots \in \Gamma$. Then

$$\|\alpha_\i - \alpha_\i\| = \sum_{q=1}^\i c_\i e^{\lambda_\i} \| = \text{l.u.b.}\left\{\sum_{q=1}^\i |c_\i|^{1/\lambda_\i}, \; \i \geq 1\right\} \leq 2^{D+1} < \infty.$$
The following corollary follows immediately from Theorem 9 and Theorem 11.

**Corollary 3.** \((\overline{\Omega}, B)\) is the completion of \((\Omega, B)\).

**Acknowledgment**

The authors are thankful to the referees for their helpful comments and suggestions for improving the paper.

**References**


Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee, 247-667-India.
E-mail: girssfma@iitr.ernet.in