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ON A NEW TYPE OF GENERALIZED DIFFERENCE CESÀRO SEQUENCE SPACES

BY

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Abstract. In this paper we introduce the generalized difference Cesàro sequence spaces $C_\infty(\Delta^n_m)$, $O_\infty(\Delta^n_m)$, $C_p(\Delta^n_m)$, $O_p(\Delta^n_m)$ and $\ell_p(\Delta^n_m)$ for $1 \le p < \infty$. We study some topological properties of these spaces. We obtain some inclusion relations involving these sequence spaces. These notions generalize many notions on difference Cesàro sequence spaces.

1. Introduction

Throughout the paper w, ℓ_∞, ℓ_p, c and c_0 denote the spaces of *all*, *bounded*, *p-absolutely summable*, *convergent* and *null* sequences $x = (x_k)$ with complex terms respectively. The zero sequence is denoted by $\theta = (0, 0, 0, \ldots)$.

The notion of difference sequence space was introduced by Kizmaz [3], who studied the difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Colak [1] as follows :

$$Z(\Delta^n) = \{x = (x_k) \in w : (\Delta^n x_k) \in Z\},$$

for $Z = \ell_\infty, c$ and c_0, where $\Delta^n x = (\Delta^n x_k) = (\Delta^{n-1}x_k - \Delta^{n-1}x_{k+1})$ and $\Delta^0 x_k = x_k$ for all $k \in N$, or equivalent to

$$\Delta^n x_k = \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} x_{k+\nu}.$$

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Recently the idea was generlized by Tripathy and Esi [11] as follows :

Let $m \geq 0$, be a fixed integer, then

$$Z(\Delta_m) = \{x = (x_k) \in w : (\Delta_m x_k) \in Z\},$$

for $Z = \ell_\infty, c$ and c_0, where $\Delta_m x = (\Delta_m x_k) = (x_k - x_{k+m})$ and $\Delta_0 x_k = x_k$ for all $k \in N$.

They showed that the above spaces are Banach spaces, normed by

$$\|(x_k)\|_{\Delta_m} = \sum_{k=1}^{m} |x_k| + \sup_k \|\Delta_m x_k\|.$$

Ng and Lee [6] defined the Cesàro sequence sapces X_p of non-absolute type as follows :

$$x = (x_k) \in X_p \text{ if and only if } \sigma(x) \in \ell_p,\ 1 \leq p < \infty,$$

where $\sigma(x) = \Big(\frac{1}{n}\sum_{k=1}^{n} x_k\Big)_{n=1}^{\infty}$.

Orhan [7] defined the Cesàro difference sequence spaces $X_p(\Lambda)$, for $1 \leq p \leq \infty$ and studied their different properties and proved some inclusion results. He also obtained the duals of these sequence spaces.

Mursaleen, Gaur and Saifi [4] defined the second difference Cesàro sequence spaces $X_p(\Delta^2)$, for $1 \leq p \leq \infty$ and studied their different topological properties and proved some inclusion results. They also studied their dual spaces.

2. Definitions and Preliminaries

A sequence space E is said to be *solid* (or *normal*) if $(x_k) \in E$ implies $(\alpha_k x_k) \in E$ for all sequences of scalars (α_k) with $|\alpha_k| \leq 1$ for all $k \in N$.

A sequence space E is said to be *monotone* if it contains the canonical preimages of all its step spaces.

A sequence space E is said to be *convergence free* if $(y_k) \in E$ whenever $(x_k) \in E$ and $y_k = 0$ whenever $x_k = 0$.

A sequence space E is said to be *symmetric* if $(x_{\pi(k)}) \in E$ whenever $(x_k) \in E$, where $\pi(k)$ is a pemutation on N.

Let $m, n \geq 0$ be fixed integers and $1 \leq p < \infty$, then in this article we introduce the following new types of generalized difference Cesàro sequence spaces:

$$C_p(\Delta_m^n) = \{x = (x_k) : \sum_{i=1}^{\infty} |\frac{1}{i} \sum_{k=1}^{i} \Delta_m^n x_k|^p < \infty\},$$

$$C_\infty(\Delta_m^n) = \{x = (x_k) : \sup_i |\frac{1}{i} \sum_{k=1}^{i} \Delta_m^n x_k| < \infty\},$$

$$\ell_p(\Delta_m^n) = \{x = (x_k) : \sum_{k=1}^{\infty} |\Delta_m^n x_k|^p < \infty\},$$

$$O_p(\Delta_m^n) = \{x = (x_k) : \sum_{i=1}^{\infty} (|\frac{1}{i} \sum_{k=1}^{i} \Delta_m^n x_k|)^p < \infty\},$$

$$O_\infty(\Delta_m^n) = \{x = (x_k) : \sup_i \frac{1}{i} \sum_{k=1}^{i} |\Delta_m^n x_k| < \infty\},$$

where $\Delta_m^n x = (\Delta_m^n x_k) = (\Delta_m^{n-1} x_k - \Delta_m^{n-1} x_{k+m})$ and $\Delta_m^0 x_k = x_k$ for all $k \in N$. This generalized difference notion has the following binomial representation :

$$\Delta_m^n x_k = \sum_{\nu=0}^{n} (-1)^n \binom{n}{\nu} x_{k+m\nu}.$$

For $n = m = 0$, these spaces reduce to the spaces C_p and C_∞ studied by Ng and Lee [6] and the spaces O_p and O_∞ studied by Shiue [8].

For $n = m = 1$, these represent the spaces $C_p(\Delta)$, $C_\infty(\Delta)$, $O_p(\Delta)$ and $O_\infty(\Delta)$ introduced and studied by Orhan [7].

For $m = 1, n = 2$, the spaces $C_p(\Delta^2)$ and $C_\infty(\Delta^2)$ are studied by Mursaleen et al. [4].

For $m = 1$, the spaces $C_p(\Delta^n)$ and $C_\infty(\Delta^n)$ are studied by Et [2].

From the existing literature, listed in the references, we have the following result.

Lemma. (a) *Let* $1 \leq p < \infty$. *Then:*
(i) *The space* C_p *is a Banach space, normed by*

$$\|x\| = (\sum_{i=1}^{\infty} |\frac{1}{i} \sum_{k=1}^{i} x_k|^p)^{\frac{1}{p}}.$$

(ii) *The space O_p is a Banach space, normed by*

$$\|x\| = (\sum_{i=1}^{\infty} |\frac{1}{i} \sum_{k=1}^{i} x_k|^p)^{\frac{1}{p}}.$$

(iii) *The space ℓ_p is a Banach space, normed by*

$$\|x\| = (\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}}.$$

(b) (i) *The space C_∞ is a Banach space, normed by*

$$\|x\| = \sup_i |\frac{1}{i} \sum_{k=1}^{i} x_k|.$$

(ii) *The space O_∞ is a Banach space, normed by*

$$\|x\| = \sup_i \frac{1}{i} \sum_{k=1}^{i} |x_k|.$$

3. Main Results

In this section we prove the results of this article. The proof of the following result is a routine verification.

Proposition 1. *The classes of sequences $C_\infty(\Delta_m^n)$, $O_\infty(\Delta_m^n)$, $C_p(\Delta_m^n)$, $O_p(\Delta_m^n)$ and $\ell_p(\Delta_m^n)$ for $1 \le p < \infty$ are linear spaces.*

Theorem 2. *Let Z be a Banach space normed by f, then the space $Z(\Delta_m^n)$ is a Banach space normed by*

$$g(x) = \sum_{k=1}^{nm} |x_k| + f(\Delta_m^n x_k).$$

Proof. Let $(x^s)_{s=1}^{\infty}$ be a Cauchy sequence in $Z(\Delta_m^n)$, where $x^s = (x_i^s)_{i=1}^{\infty}$ for each $s \in N$. We have for a given $\varepsilon > 0$, there exists $\eta \in N$ such that

$$\begin{aligned} & g(x^s - x^t) < \varepsilon, \text{ for all } s, t \ge \eta \\ \Rightarrow & \sum_{k=1}^{nm} |x_k^s - x_k^t| + f(\Delta_m^n (x_k^s - x_k^t)) < \varepsilon, \text{ for all } s, t \ge \eta \\ \Rightarrow & \sum_{k=1}^{nm} |x_k^s - x_k^t| < \varepsilon, \end{aligned}$$

and $f(\Delta_m^n(x_k^s - x_k^t)) < \varepsilon$, for all $s, t \geq \eta$.

Hence $|x_k^s - x_k^t| < \varepsilon$ for all $k = 1, 2, ..., nm$.

$\Rightarrow$ (x_k^s) is a Cauchy sequence for all $k = 1, 2, ..., nm$ in $\mathbf{C}$, the fileld of complex numbers.

Hence (x_k^s) converges in $\mathbf{C}$ for all $k = 1, 2, \ldots, nm$. Let $\lim_{s\to\infty} x_k^s = x_k$ for all $k = 1, 2, \ldots, nm$.

Next we have $f(\Delta_m^n(x_k^s - x_k^t)) < \varepsilon$, for all $s, t \geq \eta$.

$\Rightarrow (\Delta_m^n x_k^s)$ is a Cauchy sequence in Z, which is complete.

Hence $(\Delta_m^n x_k^s)$ converges for each $k \in N$. Let $\lim_{s\to\infty} \Delta_m^n x_k^s = y_k$ for each $k \in N$.

Let $k = 1$, we have

$$\lim_{s\to\infty} \Delta_m^n x_1^s = \lim_{s\to\infty} \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} x_{1+m\nu} = y_1. \tag{1}$$

We have

$$\lim_{s\to\infty} x_k^s = x_k, \text{ for } k = 1 + m\nu, \text{ for } \nu = 0, 1, 2, \ldots, n-1. \tag{2}$$

Thus from (1) and (2) we have $\lim_{s\to\infty} x_{1+nm}^s$ exists. Let $\lim_{s\to\infty} x_{1+nm}^s = x_{1+nm}$. Proceeding in this way inductively, we have $\lim_{s\to\infty} x_k^s = x_k$ exists for each $k \in N$.

Now we have for all $s, t \geq \eta$,

$$\sum_{k=1}^{nm} |x_k^s - x_k^t| + f(\Delta_m^n(x_k^s - x_k^t)) < \varepsilon$$
$$\Rightarrow \lim_{s\to\infty} \Big\{ \sum_{k=1}^{nm} |x_k^s - x_k^t| + f(\Delta_m^n(x_k^s - x_k^t)) \Big\} \leq \varepsilon, \text{ for all } s \geq \eta$$
$$\Rightarrow \sum_{k=1}^{nm} |x_k^s - x_k| + f(\Delta_m^n(x_k^s - x_k)) \leq \varepsilon, \text{ for all } s \geq \eta$$
$$\Rightarrow g(x^s - x) \leq \varepsilon, \text{ for all } s \geq \eta.$$

Hence $(x^s - x) \in Z(\Delta_m^n)$. Since $Z(\Delta_m^n)$ is a linear space, so we have for all $s \geq \eta$,

$$x = x^s - (x^s - x) \in Z(\Delta_m^n).$$

Hence $Z(\Delta_m^n)$ is complete and as such is a Banach space.

The proof of the following result is a consequence of the above result and lemma.

Corollary 3. (a) *Let* $1 \leq p < \infty$. *Then:*
(i) *The space* $C_p(\Delta_m^n)$ *is a Banach space, normed by*

$$\|x\| = \sum_{k=1}^{nm} |x_k| + (\sum_{i=1}^{\infty} |\frac{1}{i} \sum_{k=1}^{i} \Delta_m^n x_k|^p)^{\frac{1}{p}}.$$

(ii) *The space* $O_p(\Delta_m^n)$ *is a Banach space, normed by*

$$\|x\| = \sum_{k=1}^{nm} |x_k| + (\sum_{i=1}^{\infty} |\frac{1}{i} \sum_{k=1}^{i} \Delta_m^n x_k|^p)^{\frac{1}{p}}.$$

(iii) *The space* $\ell_p(\Delta_m^n)$ *is a Banach space, normed by*

$$\|x\| = \sum_{k=1}^{nm} |x_k| + (\sum_{k=1}^{\infty} |\Delta_m^n x_k|^p)^{\frac{1}{p}}.$$

(b) (i) *The space* $C_\infty(\Delta_m^n)$ *is a Banach space, normed by*

$$\|x\| = \sum_{k=1}^{nm} |x_k| + \sup_i |\frac{1}{i} \sum_{k=1}^{i} \Delta_m^n x_k|.$$

(ii) *The space* $O_\infty(\Delta_m^n)$ *is a Banach space, normed by*

$$\|x\| = \sum_{k=1}^{nm} |x_k| + \sup_i \frac{1}{i} \sum_{k=1}^{i} |\Delta_m^n x_k|.$$

Theorem 4. *The spaces* $C_\infty(\Delta_m^n)$, $O_\infty(\Delta_m^n)$, $C_p(\Delta_m^n)$, $O_p(\Delta_m^n)$ *and* $\ell_p(\Delta_m^n)$ *for* $1 \leq p < \infty$ *are not monotone and as such are not solid for* $m, n \geq 1$.

Proof. The proof follows from the following example.

Example 1. Let $n = 2$ and $m = 3$. Then $\Delta_3^2 x_k = x_k - 2x_{k+3} + x_{k+6}$, for all $k \in N$. Consider the J^{th} step space of a sequence space E defined as, for $(x_k), (y_k) \in E_J$ implies that $y_k = x_k$ for k odd and $y_k = 0$ for k even. Consider the sequence (x_k) defined as $x_k = 1$ for all $k \in N$. Then $(x_k) \in Z(\Delta_3^2)$ for

$Z = C_p,\ O_p,\ \ell_p,\ C_\infty$ and O_∞, but its J^{th} canonical pre-image does not belong to $Z(\Delta_3^2)$ for $Z = C_p,\ O_p,\ \ell_p,\ C_\infty$ and O_∞. Hence the spaces are not monotone and as such are not solid.

Remark. For $m = 0$ or $n = 0$, the spaces C_p and C_∞ are neither solid nor monotone, where as the spaces O_p, ℓ_p and O_∞ are solid and hence are monotone.

Theorem 5. *The spaces $Z(\Delta_m^n)$, for $Z = C_p,\ O_p,\ \ell_p,\ C_\infty$ and O_∞ are not convergence free.*

Proof. The proof follows from the following example.

Example 2. Let $n = 2$ and $m = 2$. Then $\Delta_2^2 x_k = x_k - 2x_{k+2} + x_{k+4}$, for all $k \in N$. Consider the sequences (x_k) and (y_k) defined as $x_k = 4$ for all $k \in N$ and $y_k = k^2$ for all $k \in N$. Then $(x_k) \in Z(\Delta_m^n)$ but $(y_k) \notin Z(\Delta_m^n)$, for $Z = C_p, O_p, \ell_p, C_\infty$ and O_∞. Hence the spaces $Z(\Delta_m^n)$, for $Z = C_p, O_p, \ell_p, C_\infty$ and O_∞ are not convergence free.

Theorem 6. *The spaces $Z(\Delta_m^n)$, for $Z = C_p,\ O_p,\ \ell_p,\ C_\infty$ and O_∞ are not symmetric.*

Proof. The proof follows from the following example.

Example 3. Let $n = 2$ and $m = 3$. Then $\Delta_3^2 x_k = x_k - 2x_{k+3} + x_{k+6}$ for all $k \in N$. Consider the sequence (x_k) defined as $x_k = k$, for all $k \in n$. Then $\Delta_3^2 x_k = 0$ for all $k \in N$. Hence $(x_k) \in Z(\Delta_3^2)$, for $Z = C_p,\ O_p,\ \ell_p,\ C_\infty$ and O_∞. Consider the rearranged sequence, (y_k) of (x_k) defined as

$$(y_k) = (x_1, x_2, x_4, x_3, x_9, x_5, x_{16}, x_6, x_{25}, x_7, x_{36}, x_8, x_{49}, x_{10}, \ldots).$$

Then $(y_k) \notin Z(\Delta_3^2)$. Hence the spaces are not symmetric.

The proofs of the following results are straightforward.

Theorem 7. (a) $\ell_p(\Delta_m^n) \subset O_p(\Delta_m^n) \subset C_p(\Delta_m^n)$ *and the inclusions are strict.*
(b) $Z(\Delta_m^{n-1}) \subset Z(\Delta_m^n)$ (*in general* $Z(\Delta_m^i) \subset Z(\Delta_m^n)$, *for* $i = 1, 2, 3, \ldots, n-1$), *for* $Z = C_p,\ O_p,\ \ell_p,\ C_\infty$ *and* O_∞.
(c) $O_\infty(\Delta_m^n) \subset C_\infty(\Delta_m^n)$ *and the inclusion is strict.*

Theorem 8. (a) *If* $1 \leq p < q$, *then*

(i) $C_p(\Delta_m^n) \subset C_q(\Delta_m^n)$.

(ii) $\ell_p(\Delta_m^n) \subset \ell_q(\Delta_m^n)$.

(b) $C_p \subset C_p(\Delta_m^n)$, *for all* $m \geq 1$ *and* $n \geq 1$.

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