A NEW OSTROWSKI TYPE INEQUALITY FOR DOUBLE INTEGRALS

BY

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Abstract. The main aim of this note is to establish a new Ostrowski type inequality for double integrals by using a fairly elementary analysis. The discrete analogue of the result is also given.

1. Introduction

The classical integral inequality established by A. M. Ostrowski [6] in 1938 can be stated as follows (see also [5, p.469]).

Let \( f : [a,b] \rightarrow \mathbb{R} \) be continuous on \( [a,b] \) and differentiable on \((a,b)\) whose derivative \( f' : (a,b) \rightarrow \mathbb{R} \) is bounded on \((a,b)\), i.e., \( \|f'\|_\infty = \sup_{t \in (a,b)} |f'(t)| < \infty \), then

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{1}{4} + \frac{\left( x - \frac{a+b}{2} \right)^2}{(b-a)^2} (b-a)\|f'\|_\infty, \tag{1.1}
\]

for all \( x \in [a,b] \).

In [2], Barnett and Dragomir proved the following Ostrowski type inequality for double integrals.

Let \( f : [a,b] \times [c,d] \rightarrow \mathbb{R} \) be continuous on \([a,b] \times [c,d]\), \( f''_{x,y} = \frac{\partial^2 f}{\partial x \partial y} \) exists on \((a,b) \times (c,d)\) and is bounded, that is,

\[
\|f''_{x,y}\|_\infty = \sup_{(x,y) \in (a,b) \times (c,d)} \left| \frac{\partial^2 f(x,y)}{\partial x \partial y} \right| < \infty,
\]

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then we have the inequality
\[
\left| \int_a^b \int_c^d f(s, t)dt ds - (b - a) \int_c^d f(x, t)dt \right|
-(d - c) \int_a^b f(s, y)ds + (b - a)(d - c)f(x, y) \leq \left[ \frac{(b - a)^2}{4} + \left( x - \frac{a + b}{2} \right)^2 \right] \left[ \frac{(d - c)^2}{4} + \left( y - \frac{c + d}{2} \right)^2 \right] \|f_{x,y}\|_{\infty},
\] (1.2)
for all \((x, y) \in [a, b] \times [c, d]\).

In [2], the inequality (1.2) is established by the use of integral identity involving Peano kernels. In [8] Pachpatte obtained an inequality in the view of (1.2) by using elementary analysis. The interested reader is also referred to [1, 3, 4, 7, 9, 10] for Ostrowski type inequalities in several independent variables. The object of this note is to establish a new Ostrowski type inequality for double integrals involving two functions and their partial derivatives. An interesting feature of our result is that, it is presented in an elementary way and in the special case, we recapture the inequality (1.2) given in [2]. The discrete version of the main result is also given.

2. Statement of Results

In what follows \(R\) denotes the set of real numbers, and \([a, b], [c, d]\) \((a < b, c < d)\) are the given subsets of \(R\). If \(z(x, y)\) is a differentiable function defined on \([a, b] \times [c, d]\), then its partial derivatives are denoted by \(D_1z(x, y) = \frac{\partial}{\partial x}z(x, y)\), \(D_2z(x, y) = \frac{\partial}{\partial y}z(x, y)\), \(D_2D_1z(x, y) = \frac{\partial^2}{\partial y\partial x}z(x, y)\). Let \(N\) be the set of natural numbers and \(A = \{1, 2, \ldots, k + 1\}, B = \{1, 2, \ldots, r + 1\}, (k, r \in N), H = A \times B\). For a function \(w : H \rightarrow \mathbb{R}\) we define the difference operators \(\Delta_1w(m, n) = w(m + 1, n) - w(m, n), \Delta_2w(m, n) = w(m, n + 1) - w(m, n), \Delta_2\Delta_1w(m, n) = \Delta_2(\Delta_1w(m, n)).\) We use the usual convention that, empty sum is taken to be 0.

Our main result is given in the following theorem.

**Theorem 1.** Let \(f, g : [a, b] \times [c, d] \rightarrow \mathbb{R}\) be continuous mappings such that \(D_2D_1f, D_2D_1g\) exist and are continuous on \([a, b] \times [c, d]\), then
\[
\left| f(x, y)g(x, y) - \frac{1}{2}g(x, y) \left[ \frac{1}{b - a} \int_a^b f(s, y)ds + \frac{1}{d - c} \int_c^d f(x, t)dt \right] \right|
\]
\[ - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(s, t) dt ds \] 
\[ - \frac{1}{2} f(x, y) \left( \frac{1}{b-a} \int_a^b g(s, y) ds + \frac{1}{d-c} \int_c^d g(x, t) dt \right) \] 
\[ - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d g(s, t) dt ds \] 
\[ \leq \frac{1}{2(b-a)(d-c)} \int_a^b \int_c^d \left( \|g(x, y)\| \int_s^y D_2 D_1 f(\sigma, \tau) d\tau d\sigma \right) \] 
\[ + \|f(x, y)\| \int_s^y \int_t^y D_2 D_1 g(\sigma, \tau) d\tau d\sigma \] 
\[ dy dx, \]  
(2.1)

for all \((x, y) \in [a, b] \times [c, d]\).

**Remark 1.** By taking \(g(x, y) = 1\) and hence \(D_2 D_1 g(x, y) = 0\) in Theorem 1, we get

\[ \|f(x, y) - \frac{1}{b-a} \int_a^b f(s, y) ds - \frac{1}{d-c} \int_c^d f(x, t) dt \] 
\[ - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(s, t) dt ds \| \] 
\[ \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \left( \|g(x, y)\| \int_s^y D_2 D_1 f(\sigma, \tau) d\tau d\sigma \right) \] 
\[ + \|f(x, y)\| \int_s^y \int_t^y D_2 D_1 g(\sigma, \tau) d\tau d\sigma \] 
\[ dy dx, \]  
(2.2)

for all \((x, y) \in [a, b] \times [c, d]\). Further, if we assume that \(D_2 D_1 f(x, y)\) is bounded on \((a, b) \times (c, d)\), i.e.,

\[ \|D_2 D_1 f\|_\infty = \sup_{(x, y) \in (a, b) \times (c, d)} |D_2 D_1 f(x, y)| < \infty, \]

then after rewriting (2.2) and by elementary calculations we recapture the inequality (1.2) given by Barnett and Dragomir in [2].

The discrete version of Theorem 1 is embodied in the following theorem.

**Theorem 2.** Let \(f, g : H \rightarrow R\) and \(\Delta_2 \Delta_1 f(m, n), \Delta_2 \Delta_1 g(m, n)\) exist on \(H\), then

\[ \left| f(m, n) g(m, n) - \frac{1}{2} g(m, n) \left[ \frac{1}{k} \sum_{s=1}^k f(s, n) + \frac{1}{r} \sum_{t=1}^r f(m, t) - \frac{1}{k r} \sum_{s=1}^k \sum_{t=1}^r f(s, t) \right] \right| \]
\[ - \frac{1}{2} f(m, n) \left[ \frac{1}{r} \sum_{s=1}^k g(s, n) + \frac{1}{r} \sum_{t=1}^r g(m, t) - \frac{1}{k r} \sum_{s=1}^k \sum_{t=1}^r g(s, t) \right] \]
\begin{equation}
\begin{aligned}
&\leq \frac{1}{2kr} \sum_{s=1}^{k} \sum_{t=1}^{r} \left[ |g(m, n)| \left| \sum_{\sigma=s}^{m-1} \sum_{\tau=t}^{n-1} \Delta_2 \Delta_1 f(\sigma, \tau) \right| \\
&+ |f(m, n)| \left| \sum_{\sigma=s}^{m-1} \sum_{\tau=t}^{n-1} \Delta_2 \Delta_1 g(\sigma, \tau) \right| \right],
\end{aligned}
\tag{2.3}
\end{equation}

for all \((m, n) \in H\).

**Remark 2.** In the special case, if we take \(g(m, n) = 1\) and hence \(\Delta_2 \Delta_1 g(m, n) = 0\) in Theorem 2, then by simple calculations we get

\begin{equation}
\begin{aligned}
&\left| f(m, n) - \left[ \frac{1}{k} \sum_{s=1}^{k} f(s, n) + \frac{1}{r} \sum_{t=1}^{r} f(m, t) - \frac{1}{kr} \sum_{s=1}^{k} \sum_{t=1}^{r} f(s, t) \right] \right| \\
&\leq \frac{1}{kr} \sum_{s=1}^{k} \sum_{t=1}^{r} \left| \sum_{\sigma=s}^{m-1} \sum_{\tau=t}^{n-1} \Delta_2 \Delta_1 f(\sigma, \tau) \right|,
\end{aligned}
\tag{2.4}
\end{equation}

for all \((m, n) \in H\).

3. Proofs of Theorems 1 and 2

From the hypothesis of Theorem 1, it is easy to observe that, the following identities hold for \((x, y), (s, t) \in [a, b] \times [c, d]\):

\begin{align}
&f(x, y) - f(s, y) - f(x, t) + f(s, t) = \int_{s}^{x} \int_{t}^{y} D_2 D_1 f(\sigma, \tau) d\tau d\sigma, \tag{3.1} \\
g(x, y) - g(s, y) - g(x, t) + g(s, t) = \int_{s}^{x} \int_{t}^{y} D_2 D_1 g(\sigma, \tau) d\tau d\sigma. \tag{3.2}
\end{align}

Multiplying both sides of (3.1) and (3.2) by \(g(x, y)\) and \(f(x, y)\) respectively and adding the resulting identities we have

\begin{align}
2f(x, y)g(x, y) - g(x, y)[f(s, y) + f(x, t) - f(s, t)] \\
- f(x, y)[g(s, y) + g(x, t) - g(s, t)] \\
= g(x, y) \int_{s}^{x} \int_{t}^{y} D_2 D_1 f(\sigma, \tau) d\tau d\sigma + f(x, y) \int_{s}^{x} \int_{t}^{y} D_2 D_1 g(\sigma, \tau) d\tau d\sigma. \tag{3.3}
\end{align}

Integrating both sides of (3.3) with respect to \((s, t)\) over \([a, b] \times [c, d]\) and rewriting we have

\begin{equation}
f(x, y)g(x, y) - \frac{1}{2} g(x, y) \left[ \frac{1}{b - a} \int_{a}^{b} f(s, y) ds + \frac{1}{d - c} \int_{c}^{d} f(x, t) dt \right]
\end{equation}
From (3.4) and using the properties of modulus, it is easy to observe that the required inequality in (2.1) holds. The proof of Theorem 1 is complete.

From the hypotheses of Theorem 2, by simple computation we have the following identities for \((m, n), (s, t) \in H:\)

\[ f(m, n) - f(s, n) - f(m, t) + f(s, t) = \sum_{\sigma = s}^{m-1} \sum_{\tau = t}^{n-1} \Delta_2 \Delta_1 f(\sigma, \tau), \quad (3.5) \]

\[ g(m, n) - g(s, n) - g(m, t) + g(s, t) = \sum_{\sigma = s}^{m-1} \sum_{\tau = t}^{n-1} \Delta_2 \Delta_1 g(\sigma, \tau). \quad (3.6) \]

Multiplying both sides of (3.5) and (3.6) by \(g(m, n)\) and \(f(m, n)\) respectively and adding the resulting identities we have

\[ 2f(m, n)g(m, n) - g(m, n)[f(s, n) + f(m, t) - f(s, t)] \]

\[ -f(m, n)[g(s, n) + g(m, t) - g(s, t)] \]

\[ = g(m, n) \sum_{\sigma = s}^{m-1} \sum_{\tau = t}^{n-1} \Delta_2 \Delta_1 f(\sigma, \tau) + f(m, n) \sum_{\sigma = s}^{m-1} \sum_{\tau = t}^{n-1} \Delta_2 \Delta_1 g(\sigma, \tau). \quad (3.7) \]

Summing both sides of (3.7) first with respect to \(t\) from 1 to \(r\) and then with respect to \(s\) from 1 to \(k\) and rewriting we have

\[ f(m, n)g(m, n) - \frac{1}{2} g(m, n) \left[ \frac{1}{k} \sum_{s = 1}^{k} f(s, n) + \frac{1}{r} \sum_{t = 1}^{r} f(m, t) - \frac{1}{kr} \sum_{s = 1}^{k} \sum_{t = 1}^{r} f(s, t) \right] \]

\[ -\frac{1}{2} f(m, n) \left[ \frac{1}{k} \sum_{s = 1}^{k} g(s, n) + \frac{1}{r} \sum_{t = 1}^{r} g(m, t) - \frac{1}{kr} \sum_{s = 1}^{k} \sum_{t = 1}^{r} g(s, t) \right] \]

\[ = \frac{1}{2kr} \sum_{s = 1}^{k} \sum_{t = 1}^{r} \left[ g(m, n) \sum_{\sigma = s}^{m-1} \sum_{\tau = t}^{n-1} \Delta_2 \Delta_1 f(\sigma, \tau) + f(m, n) \sum_{\sigma = s}^{m-1} \sum_{\tau = t}^{n-1} \Delta_2 \Delta_1 g(\sigma, \tau) \right]. \quad (3.8) \]
From (3.8) and using the properties of nodulus we get the desired inequality in (2.3). The proof of Theorem 2 is complete.

References


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