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CHARACTERIZATIONS AND APPLICATIONS OF γ-OPEN SETS

BY

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Abstract. The notion of semi-convergence of filters was introduced by Latif [1] who investigated some characterizations related to semi-open continuous functions. In the spirit of Latif [1], Min [4] used the idea of semi-convergence of filters to introduce a new class of sets, called γ-open sets, and the notions of γ-closure, γ-interior and γ-continuity and investigated some properties. In this paper we continue to explore further properties of these notions as well as characterizations of γ-open sets. We also introduce and study topological properties of γ-derived, γ-border, γ-frontier, and γ-exterior of a set using the concept of γ-open sets.

1. Introduction

The notion of γ-open set (originally called γ-sets) in topological spaces was introduced by Min [4]. For thses sets, we introduce the notions of γ-derived, γ-border, γ-frontier and γ-exterior of a set and show that some of their properties are analogous to those for open sets. Also, we give some additional properties of γ-closure. Throughout this paper, (X, τ) (simply X) always mean topological space on which no separation axioms are assumed unless explicitly stated. Let S be a subset of X. The closure (resp., interior) of S will be denoted by $Cl(S)$ (resp., $Int(S)$).

A subset S of X is called a semi-open set (Levine [2]) (resp., α-open set) (Njastad [5]) if $S \subseteq Cl[Int(S)]$ (resp., $S \subseteq Int[Cl(Int(S))]$). The complement of

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a semi-open set (resp., α-open set) is called a semi-closed set (resp., α-closed set). The family of all semi-open sets (resp., α-open sets) in a topological space (X, τ) will be denoted by $SO(X)$ (resp., τ^{α}). A subset $M(x)$ of a space X is called a semi-neighborhood of a point $x \in X$ if there exists a semi-open set S such that $x \in S \subseteq M(x)$. In [1] Latif introduced the notion of semi-convergence of filters and investigated some characterizations related to semi-continuous functions. Now, we recall the concept of semi-convergence of filters. Let $S(x) = \{A \in SO(X) : x \in A\}$ and $S_x = \{A \subseteq X :$ there exists $\mu \subseteq S(x)$ such that μ is finite and $\cap \mu \subseteq A\}$. Then, S_x is called the semi-neighborhood filter at x. For any filter Γ on X, we say that Γ semi-converges to x if and only if Γ is finer than the semi-neighborhood filter at x.

A subset U of X is called a γ-open set if whenever a filter Γ semi-converges to x and $x \in U$, $U \in \Gamma$. The complement of a γ-open set is called a γ-closed set. The intersection of all γ-closed sets containing A is called the γ-closure of A, denoted by $Cl_{\gamma}(A)$. A subset A is also γ-closed if and only if $A = Cl_{\gamma}(A)$. We denote the family of all γ-open sets of (X, τ) by τ^{γ}. It is shown in [4] that τ^{γ} is a topology on X. In a topological space (X, τ), it is always true that $\tau \subseteq \tau^{\alpha} \subseteq SO(X, \tau) \subseteq \tau^{\gamma}$.

2. Applications of γ-Open Sets

Definition 2.1. Let A be a subset of a space X. A point $x \in A$ is called a γ-limit point of A if for each γ-open set U containing x, $U \cap (A - \{x\}) \neq \varnothing$. The set of all γ-limit points of A is called a γ-derived set of A and is denoted by $D_{\gamma}(A)$.

Theorem 2.2. *For subsets A, B of a space X, the following statements hold:*
(1) $D_{\gamma}(A) \subseteq D(A)$, *where $D(A)$ is the derived set of A.*
(2) *If $A \subseteq B$, then $D_{\gamma}(A) \subseteq D_{\gamma}(B)$.*
(3) $D_{\gamma}(A) \cup D_{\gamma}(B) \subseteq D_{\gamma}(A \cup B)$ *and* $D_{\gamma}(A \cap B) \subseteq D_{\gamma}(A) \cap D_{\gamma}(B)$.
(4) $[D_{\gamma}(D_{\gamma}(A)) - A] \subseteq D_{\gamma}(A)$.
(5) $D_{\gamma}[A \cup D_{\gamma}(A)] \subseteq A \cup D_{\gamma}(A)$.

Proof. (1) It suffices to observe that every open set is γ-open. (2) Obvious. (3) Follows by (2).

(4) If $x \in [D_\gamma(D_\gamma(A)) - A]$ and U is a γ-open set containig x, then $U \cap [D_\gamma(A) - \{x\}] \neq \varnothing$. Let $y \in U \cap [D_\gamma(A) - \{x\}]$. Then, since $y \in D_\gamma(A)$ and $y \in U$, so $U \cap (A - \{y\}) \neq \varnothing$. Let $z \in U \cap (A - \{y\}) \neq \varnothing$. Then, $z \neq x$ for $z \in A$ and $x \notin A$. Hence, $U \cap (A - \{x\}) \neq \varnothing$. Therefore, $x \in D_\gamma(A)$.

(5) Let $x \in D_\gamma[A \cup D_\gamma(A)]$. If $x \in A$, the result is obvious. So, let $x \in [D_\gamma(A \cup D_\gamma(A)) - A]$. Then, for any γ-open set U containing x, $U \cap [(A \cup D_\gamma(A)) - \{x\}] \neq \varnothing$. Thus, $U \cap (A - \{x\}) \neq \varnothing$ or $U \cap (D_\gamma(A) - \{x\}) \neq \varnothing$. Now, it follows similarly from (4) that $U \cap (A - \{x\}) \neq \varnothing$. Hence, $x \in D_\gamma(A)$. Therefore, in any case, $D_\gamma[A \cup D_\gamma(A)] \subseteq [A \cup D_\gamma(A)]$.

In general, the converse of (1) may not be true and the equality does not hold in (3) of Theorem 2.2.

Example 2.3. Let $X = \{a, b, c\}$ with topology $\tau = \{\varnothing, \{a\}, X\}$. Thus, $\tau^\gamma = \{\varnothing, \{a\}, \{a, b\}, \{a, c\}, X\}$. Consider the following:

(i) $A = \{c\}$. Then, $D(A) = \{b\}$ and $D_\gamma(A) = \varnothing$. Hence, $D(A) \not\subset D_\gamma(A)$.

(ii) $C = \{a\}$ and $E = \{b, c\}$. Then, $D_\gamma(C) = \{b, c\}$ and $D_\gamma(E) = \varnothing$.
Hence, $D_\gamma(C \cup E) \neq D_\gamma(C) \cup D_\gamma(E)$.

Theorem 2.4. *For any subset A of a space X, $Cl_\gamma(A) = A \cup D_\gamma(A)$.*

Proof. Since $D_\gamma(A) \subseteq Cl_\gamma(A)$, $A \cup D_\gamma(A) \subseteq Cl_\gamma(A)$. On the other hand, let $x \in Cl_\gamma(A)$. If $x \in A$, then the proof is complete. If $x \notin A$, each γ-open set U containing x intersects A at a point distinct from x; so $x \in D_\gamma(A)$. Thus, $Cl_\gamma(A) \subseteq A \cup D_\gamma(A)$, which completes the proof.

Corollary 2.5. *A subset A is γ-closed if and only if it contains the set of its γ-limit points.*

Definition 2.6.(Min [4]) A point $x \in X$ is said to be a γ-interior point of A if there exists a γ-open set U containing x such that $U \subseteq A$. The set of all γ-interior points of A is called the γ-interior of A and denoted by $Int_\gamma(A)$.

Theorem 2.7. *For subsets A, B of a space X, the following statements are true:*

(1) $Int_\gamma(A)$ *is the largest* γ*-open set contained in* A;
(2) A *is* γ*-open if and only if* $A = Int_\gamma(A)$;
(3) $Int_\gamma[Int_\gamma(A)] = Int_\gamma(A)$;
(4) $Int_\gamma(A) = A - D_\gamma(X - A)$;
(5) $X - Int_\gamma(A) = Cl_\gamma(X - A)$;
(6) $X - Cl_\gamma(A) = Int_\gamma(X - A)$;
(7) $A \subseteq B$, *then* $Int_\gamma(A) \subseteq Int_\gamma(B)$;
(8) $Int_\gamma(A) \cup Int_\gamma(B) \subseteq Int_\gamma(A \cup B)$;
(9) $Int_\gamma(A \cap B) \subseteq Int_\gamma(A) \cap Int_\gamma(B)$.

Proof. (4) If $x \in [A - D_\gamma(X - A)]$, then $x \notin D_\gamma(X - A)$ and so there exists a γ-open set U containing x such that $U \cap (X - A) = \varnothing$. Then $x \in U \subseteq A$ and hence $x \in Int_\gamma(A)$, that is, $[A - D_\gamma(X - A)] \subseteq Int_\gamma(A)$. On the other hand, if $x \in Int_\gamma(A)$, then $x \notin D_\gamma(X - A)$ since $Int_\gamma(A)$ is γ-open and $Int_\gamma(A) \cap (X - A) = \varnothing$. Hence, $Int_\gamma(A) = A - D_\gamma(X - A)$.

(5) $X - Int_\gamma(A) = X - [A - D_\gamma(X - A)] = (X - A) \cup D_\gamma(X - A) = Cl_\gamma(X - A)$.

In general, the converse of (8) may not be true.

Example 2.8. Let $X = \{a, b, c, d\}$ with topology $\tau = \{\varnothing, \{a\}, \{a, b\}, X\}$. Thus, $\tau^\gamma = \{\varnothing, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X\}$. Let $A = \{a, c\}$, $B = \{b, c\}$. Then $Int_\gamma(A) = \{a, c\}$, $Int_\gamma(B) = \varnothing$ and $Int_\gamma(A \cup B) = \{a, b, c\}$. Hence, $Int_\gamma(A \cup B) \neq Int_\gamma(A) \cup Int_\gamma(B)$.

Definition 2.9. $Bd_\gamma(A) = A - Int_\gamma(A)$ is called the γ-border of A.

Theorem 2.10. *For a subset* A *of a space* X*, the following statements hold:*
(1) $Bd_\gamma(A) \subseteq Bd(A)$ *where* $Bd(A)$ *denotes the border of* A;
(2) $A = Int_\gamma(A) \cup Bd_\gamma(A)$;
(3) $Int_\gamma(A) \cap Bd_\gamma(A) = \varnothing$;
(4) A *is a* γ*-open set if and only if* $Bd_\gamma(A) = \varnothing$;
(5) $Bd_\gamma[Int_\gamma(A)] = \varnothing$;
(6) $Int_\gamma[Bd_\gamma(A)] = \varnothing$;
(7) $Bd_\gamma[Bd_\gamma(A)] = Bd_\gamma(A)$;
(8) $Bd_\gamma(A) = A \cap Cl_\gamma(X - A)$;

(9) $Bd_\gamma(A) = D_\gamma(X - A)$.

Proof. (6) If $x \in Int_\gamma[Bd_\gamma(A)]$, then $x \in Bd_\gamma(A)$. On the other hand, since $Bd_\gamma(A) \subseteq A$, $x \in Int_\gamma[Bd_\gamma(A)] \subseteq Int_\gamma(A)$. Hence, $x \in Int_\gamma(A) \cap Bd_\gamma(A)$, which contradicts (3). Thus, $Int_\gamma[Bd_\gamma(A)] = \varnothing$.

(8) $Bd_\gamma(A) = A - Int_\gamma(A) = A - [X - Cl_\gamma(X - A)] = A \cap Cl_\gamma(X - A)$.

(9) $Bd_\gamma(A) = A - Int_\gamma(A) = A - [A - D_\gamma(X - A)] = D_\gamma(X - A)$.

In general, the converse of Theorem 2.10 (1) may not be true.

Example 2.11. Consider the topological space (X, τ) given in Example 2.3. If $A = \{a, b\}$, then $Bd_\gamma(A) = \varnothing$ and $Bd(A) = \{b\}$. Hence, $Bd(A) \not\subseteq Bd_\gamma(A)$.

Definition 2.12. $Fr_\gamma(A) = Cl_\gamma(A) - Int_\gamma(A)$ is called the γ- frontier of A.

Theorem 2.13. *For a subset A of a space X, the following statements hold:*

(1) $Fr_\gamma(A) \subseteq Fr(A)$ *where $Fr(A)$ denotes the frontier of A;*
(2) $Cl_\gamma(A) = Int_\gamma(A) \cup Fr_\gamma(A)$;
(3) $Int_\gamma(A) \cap Fr_\gamma(A) = \varnothing$;
(4) $Bd_\gamma(A) \subseteq Fr_\gamma(A)$;
(5) $Fr_\gamma(A) = Bd_\gamma(A) \cup D_\gamma(A)$;
(6) *A is a γ-open set if and only if* $Fr_\gamma(A) = D_\gamma(A)$;
(7) $Fr_\gamma(A) = Cl_\gamma(A) \cap Cl_\gamma(X - A)$;
(8) $Fr_\gamma(A) = Fr_\gamma(X - A)$;
(9) $Fr_\gamma(A)$ *is γ-closed.*
(10) $Fr_\gamma[Fr_\gamma(A)] \subseteq Fr_\gamma(A)$;
(11) $Fr_\gamma[Int_\gamma(A)] \subseteq Fr_\gamma(A)$;
(12) $Fr_\gamma[Cl_\gamma(A)] \subseteq Fr_\gamma(A)$;
(13) $Int_\gamma(A) = A - Fr_\gamma(A)$.

Proof. (2) $Int_\gamma(A) \cup Fr_\gamma(A) = Int_\gamma(A) \cup [Cl_\gamma(A) - Int_\gamma(A)] = Cl_\gamma(A)$.

(3) $Int_\gamma(A) \cap Fr_\gamma(A) = Int_\gamma(A) \cap [Cl_\gamma(A) - Int_\gamma(A)] = \varnothing$.

(5) Since $Int_\gamma(A) \cup Fr_\gamma(A) = Int_\gamma(A) \cup Bd_\gamma(A) \cup D_\gamma(A)$, $Fr_\gamma(A) = Bd_\gamma(A) \cup D_\gamma(A)$.

(7) $Fr_\gamma(A) = Cl_\gamma(A) - Int_\gamma(A) = Cl_\gamma(A) \cap Cl_\gamma(X - A)$.

(9) $Cl_\gamma[Fr_\gamma(A)] = Cl_\gamma[Cl_\gamma(A) \cap Cl_\gamma(X-A)] \subseteq Cl_\gamma[Cl_\gamma(A)] \cap Cl_\gamma[Cl_\gamma(X-A)] = Cl_\gamma(A) \cap Cl_\gamma(X-A) = Fr_\gamma(A)$.
Hence, $Fr_\gamma(A)$ is γ-closed.

(10) $Fr_\gamma[Fr_\gamma(A)] = Cl_\gamma[Fr_\gamma(A)] \cap Cl_\gamma[X - Fr_\gamma(A)] \subseteq Cl_\gamma[Fr_\gamma(A)] = Fr_\gamma(A)$.

(12) $Fr_\gamma[Cl_\gamma(A) = Cl_\gamma[Cl_\gamma(A)] - Int_\gamma[Cl_\gamma(A)] = Cl_\gamma(A) - Int_\gamma[Cl_\gamma(A)] \subseteq [Cl_\gamma(A) - Int_\gamma(A)] = Fr_\gamma(A)$.

(13) $A - Fr_\gamma(A) = A - [Cl_\gamma(A) - Int_\gamma(A)] = Int_\gamma(A)$.

The converse of (1) and (4) of Theorem 2.13 are not true in general, as shown by Example 2.14.

Example 2.14. Consider the topological space (X, τ) given in Example 2.3. If $A = \{c\}$, then $Fr(A) = \{b, c\} \not\subset \{c\} = Fr_\gamma(A)$, and if $B = \{a, b\}$, then $Fr_\gamma(B) = \{c\} \not\subset Bd_\gamma(B) = \varnothing$.

Definition 2.15.(Min [4]) A function $f : (X, \tau) \to (Y, \sigma)$ is said to be γ-continuous if $f^{-1}(V) \in \tau^\gamma$ for every $V \in \sigma$.

In the following theorem, $\#\gamma - c.$ denotes the set of points x of X for which a function $f : (X, \tau) \to (Y, \sigma)$ is not γ-continuous.

Theorem 2.16. *$\#\gamma - c.$ is identical with the union of the γ-frontiers of the inverse images of γ-open sets containing $f(x)$.*

Proof. Suppose that f is not γ-continuous at a point x of X. Then, there exists an open set $V \subseteq Y$ containing $f(x)$ such that $f(U)$ is not a subset of V for every $U \in \tau^\gamma$ containing x. Hence, we have $U \cap [X - f^{-1}(V)] \neq \varnothing$ for every $U \in \tau^\gamma$ containing x. It follows that $x \in Cl_\gamma[X - f^{-1}(V)]$. We also have $x \in f^{-1}(V)$. This means that $x \in Fr_\gamma[f^{-1}(V)]$.

Now, let f be γ-continuous at $x \in X$ and $V \subseteq Y$ any open set containing $f(x)$. Then, $x \in f^{-1}(V)$ is a γ-open set of X. Thus, $x \in Int_\gamma[f^{-1}(V)]$ and therefore $x \notin Fr_\gamma[f^{-1}(V)]$ for every open set V containing $f(x)$.

Definition 2.17. $Ext_\gamma(A) = Int_\gamma(X - A)$ is called the γ-exterior of A.

Theorem 2.18. *For a subset A of a space X, the following statements hold:*

(1) *$Ext(A) \subseteq Ext_\gamma(A)$ where $Ext(A)$ denotes the exterior of A;*
(2) *$Ext_\gamma(A)$ is γ-open;*
(3) $Ext_\gamma(A) = Int_\gamma(X - A) = X - Cl_\gamma(A)$;
(4) $Ext_\gamma[Ext_\gamma(A)] = Int_\gamma[Cl_\gamma(A)]$;
(5) *If $A \subseteq B$, then $Ext_\gamma(A) \supseteq Ext_\gamma(B)$;*
(6) $Ext_\gamma(A \cup B) \subseteq Ext_\gamma(A) \cup Ext_\gamma(B)$;
(7) $Ext_\gamma(A) \cap Ext_\gamma(B) \subseteq Ext_\gamma(A \cap B)$;
(8) $Ext_\gamma(X) = \varnothing$;
(9) $Ext_\gamma(\varnothing) = X$;
(10) $Ext_\gamma(A) = Ext_\gamma[X - Ext_\gamma(A)]$;
(11) $Int_\gamma(A) \subseteq Ext_\gamma[Ext_\gamma(A)]$;
(12) $X = Int_\gamma(A) \cup Ext_\gamma(A) \cup Fr_\gamma(A)$;
(13) $Ext_\gamma(A) \cup Ext_\gamma(A) \subseteq Ext_\gamma(A \cap B)$.

Proof.

(4) $Ext_\gamma[Ext_\gamma(A)] = Ext_\gamma[X - Cl_\gamma(A)] = Int_\gamma[X - (X - Cl_\gamma(A))] = Int_\gamma[Cl_\gamma(A)]$.
(10) $Ext_\gamma[X - Ext_\gamma(A)] = Ext_\gamma[X - Int_\gamma(X - A)] = Int_\gamma[X - (X - Int_\gamma(X - A))] = Int_\gamma[Int_\gamma(X - A)] = Int_\gamma(X - A) = Ext_\gamma(A)$.
(11) $Int_\gamma(A) \subseteq Int_\gamma[Cl_\gamma(A)] = Int_\gamma[X - Int_\gamma(X - A)] = Int_\gamma[X - Ext_\gamma(A)] = Ext_\gamma[Ext_\gamma(A)]$.
(13) $Ext_\gamma(A) \cup Ext_\gamma(B) = Int_\gamma(X - A) \cup Int_\gamma(X - B) \subseteq Int_\gamma[(X - A) \cup (X - B)] = Int_\gamma[X - (A \cap B)] = Ext_\gamma(A \cap B)$.

Example 2.19. Let $X = \{1, 2, 3, 4\}$ with topology $\tau = \{\varnothing, \{3, 4\}, X\}$. $\tau^\gamma = \{\varnothing, \{3, 4\}, \{2, 3, 4\}, \{1, 3, 4\}, X\}$. Let $A = \{1\}$ and $B = \{2\}$. Then $Ext_\gamma(A) \neq Ext(A)$, $Ext_\gamma(A \cap B) \neq Ext_\gamma(A) \cap Ext_\gamma(B)$, $Ext_\gamma(A \cup B) \neq Ext_\gamma(A) \cup Ext_\gamma(B)$, and $Ext_\gamma(A) \cup Ext_\gamma(B) \neq Ext_\gamma(A \cap B)$.

Theorem 2.20. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then the following are equivalent:*
(1) *f is γ-continuous;*
(2) *the inverse image of each closed set in Y is γ-closed in X;*
(3) *$Cl_\gamma[f^{-1}(V)] \subseteq f^{-1}[Cl(V)]$, for every $V \subseteq Y$;*
(4) *$f[Cl_\gamma(U)] \subseteq Cl[f(U)]$, for every $U \subseteq X$;*

(5) *for any point* $x \in X$ *and any open set* V *of* Y *containing* $f(x)$, *there exists* $U \in \tau^\gamma$ *such that* $x \in U$ *and* $f(U) \subseteq V$;
(6) $Bd_\gamma[f^{-1}(V)] \subseteq f^{-1}[Bd(V)]$, *for every* $V \subseteq Y$;
(7) $f[D_\gamma(U)] \subseteq Cl[f(U)]$, *for every* $U \subseteq X$;
(8) $f^{-1}[Int(B)] \subseteq Int_\gamma[f^{-1}(B)]$, *for every* $B \subseteq Y$;
(9) *whenever a filter* Γ *on* X *semi-converges to a point* x *in* X, *then* $f(\Gamma)$ *converges to* $f(x)$ *in* Y.

Proof. (1)⇒(2): Let $F \subseteq Y$ be closed. Since f is γ-continuous. $f^{-1}[Y-F] = X - f^{-1}(F)$ is γ-open. Therefore, $f^{-1}(F)$ is γ-closed in X.

(2)⇒(3): Since $Cl(V)$ is closed for every $V \subseteq Y$, then $f^{-1}[Cl(V)]$ is γ-closed. Therefore $f^{-1}[Cl(V)] = Cl_\gamma[f^{-1}(Cl(V))] \supseteq Cl_\gamma[f^{-1}(V)]$.

(3)⇒(4): Let $U \subseteq X$ and $f(U) = V$. Then $f^{-1}[Cl(V)] \supseteq Cl_\gamma[f^{-1}(V)]$. Thus $f^{-1}[Cl(f(U))] \supseteq Cl_\gamma[f^{-1}(f(U))] \supseteq Cl_\gamma(U)$ and $Cl[f(U)] \supseteq f[Cl_\gamma(U)]$.

(4)⇒(2): Let $W \subseteq Y$ be a closed set, $U = f^{-1}(W)$, then $f[Cl_\gamma(U)] \subseteq Cl[f(U)] = Cl[f(f^{-1}(W))] \subseteq Cl(W) = W$. Thus $Cl_\gamma(U) \subseteq f^{-1}[f(Cl_\gamma(U))] \subseteq f^{-1}(W) = U$. So U is γ-closed.

(2)⇒(1): Let $V \subseteq Y$ be an open set, then $Y-V$ is closed. Then $f^{-1}(Y-V) = X - f^{-1}(V)$ is γ-closed in X and hence $f^{-1}(V)$ is γ-open in X.

(1)⇒(5): Let $f : X \to Y$ be γ-continuous. For any $x \in X$ and any open set V of Y containing $f(x)$, $U = f^{-1}(V) \in \tau^\gamma$, and $f(U) = f[f^{-1}(V)] \subseteq V$.

(5)⇒(1): Let $V \in \sigma$. We prove $f^{-1}(V) \in \tau^\gamma$. Let $x \in f^{-1}(V)$. Then $f(x) \in V$ and there exists $U \in \tau^\gamma$ such that $x \in U$ and $f(x) \in f(U) \subseteq V$. Hence $x \in U \subseteq f^{-1}[f(U)] \subseteq f^{-1}(V)$. It follows that $f^{-1}(V)$ is a γ-neighborhood of each of its points. Therefore $f^{-1}(V) \in \tau^\gamma$.

(1)⇒(6): Let $V \subseteq Y$. Then we obtain $f^{-1}[Bd(V)] = f^{-1}[Cl(V) - Int(V)] = f^{-1}[Cl(V)] - f^{-1}[Int(V)]$. Since the hypothesis has been proved to be equivalent to (3). Thus by using (3), we get $Cl_\gamma[f^{-1}(V)] \subseteq f^{-1}[Cl(V)]$. Therefore, $f^{-1}[Bd(V)] \supseteq Cl_\gamma[f^{-1}(V)] - f^{-1}[Int(V)]$. Since by hypothesis $f : X \to Y$ is γ-continuous. Thus $f^{-1}[Int(V)] \in \tau^\gamma$. Hence we get $f^{-1}[Int(V)] = Int_\gamma[f^{-1}(Int(V))]$. Therefore we conclude that $f^{-1}[Bd(V)] \supseteq Cl_\gamma[f^{-1}(V)] - Int_\gamma[f^{-1}(Int(V))]$. Then this clearly implies that $f^{-1}[Bd(V) \supseteq Cl_\gamma[f^{-1}(V)] - Int_\gamma[f^{-1}(V)]$. Hence it immediately follows that $f^{-1}[Bd(V) \supseteq Bd_\gamma[f^{-1}(V)] \cup Int_\gamma[f^{-1}(V)] - Int_\gamma[f^{-1}(V)] = Bd_\gamma[f^{-1}(V)]$.

(6)⇒(1): Let $U \subseteq Y$ be an open set, $V = Y - U$. Since V is closed, $Bd_\gamma[f^{-1}(V)]$
$\subseteq f^{-1}[Bd(V)] \subseteq f^{-1}[Cl(V)] = f^{-1}(V)$. Thus $f^{-1}(V)$ is γ-closed and f is γ-continuous.

(1)⇒(7): It is obvious, since f is γ-continuous and by (4) $f[Cl_\gamma(U)] \subseteq Cl[f(U)]$ for each $U \subseteq X$. So $f[D_\gamma(U)] \subseteq Cl[f(U)]$.

(7)⇒(1): Let $U \subseteq Y$ be an open set, $V = Y - U$ and $f^{-1}(V) = W$. Then by hypothesis $f[D_\gamma(W)] \subseteq Cl[f(W)]$. Thus $f[D_\gamma(f^{-1}(V))] \subseteq Cl[f(f^{-1}(V))] \subseteq Cl(V) = V$. Then $D_\gamma[f^{-1}(V)] \subseteq f^{-1}(V)$ and $f^{-1}(V)$ is γ-closed. Therefore, f is γ-continuous.

(1)⇒(8): Let $B \subseteq Y$. Then $f^{-1}[Int(B)]$ is γ-open in X. Thus $f^{-1}[Int(B)] = Int_\gamma[f^{-1}(Int(B))] \subseteq Int_\gamma[f^{-1}(B)]$. Therefore, $f^{-1}[Int(B)] \subseteq Int_\gamma[f^{-1}(Int(B))]$.

(8)⇒(1): Let $V \subseteq Y$ be an open set. Then $f^{-1}(V) = f^{-1}[Int(V)] \subseteq Int_\gamma[f^{-1}(V)]$. Therefore, $f^{-1}(V)$ is γ-open. Hence f is γ-continuous.

(1)⇔(9): Theorem 3.3 of [4].

Remarks. (1) Every continuous function is γ-continuous.

(2) If a function $f : (X, \tau) \to (Y, \sigma)$ is γ-continuous and a function $g : (Y, \sigma) \to (Z, \vartheta)$ is γ-continuous, then $gof : (X, \tau) \to (Z, \vartheta)$ may not be γ-continuous.

(3) If a function $f : (X, \tau) \to (Y, \sigma)$ is γ-continuous and a function $g : (Y, \sigma) \to (Z, \vartheta)$ is continuous, then $gof : (X, \tau) \to (Z, \vartheta)$ is γ-continuous.

(4) Let (X, τ) and (Y, σ) be topological spaces. If $f : X \to Y$ is a function, and one of the following holds, then f is continuous.
   - (a) $f^{-1}[Int_\gamma(B)] \subseteq Int[f^{-1}(B)]$ for each $B \subseteq Y$,
   - (b) $Cl[f^{-1}(B)] \subseteq f^{-1}[Cl_\gamma(B)]$ for each $B \subseteq Y$,
   - (c) $f[Cl(A)] \subseteq Cl_\gamma[f(A)]$ for each $A \subseteq X$.

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