A METHOD FOR SEARCHING ALL THE ELEMENTS
OF A POSET \((\mathcal{M}(E), \leq)\)

BY

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Abstract. This paper presents a method for searching all the simple matroids
defined on the same set \(E\). Simultaneously, it deals with some relationships among
the matroids.

1. Introduction and Preliminaries

Matroid theory dated back to Whitney in 1935, but it did not attract the
attention of the mathematical community at that time. Until 1970’s, it started
the general development of matroid theory. Poset theory is an ancient subject
with many brilliant results and already used in the study on matroid theory.
It is well known that to find out the matroids defined on the same set is good
for not only dealing with the relationship between matroids, but also predicting
the developmental tendency of a matroid. Though many people studied various
quantitative problems about matroids (cf. [4, Chapter 16]) and some results are
obtained about asymptotic estimated of the number of non-isomorphic matroids
on an \(n\)-set (cf. [4, pp.305-308]), none of all these results was given an algorithm
or a method to earn all the matroids on an \(n\)-set. How to get every matroids
on an \(n\)-set in a constructive way, it has not been given before. How to utilize
poset theory to find out the whole elements of matroids defined on the same set?
At least to my knowledge, it has not been discussed previously. Moreover, we

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should notice that according to [4, p.54, Theorem 2] and the other discussions in [4, Ch.III], it could see the importance of simple matroids for the study on matroid theory. According to the definitions of simple matroid and matroid, we assert that it will not be difficult to find out the whole matroids defined on the same set if the whole simple matroids defined on the same set are searched out. This will be done in the near future. Based on this, all the matroids discussed in this paper are simple. To get the method finding all the simple matroids, this paper firstly builds up a binary relation \( \leq \) on \( \mathcal{M}(E) \) that is the whole simple matroids defined on a finite set \( E \), followed by discussing some properties of \( (\mathcal{M}(E), \leq) \). Finally, a method is obtained for searching out the constituent members of \( \mathcal{M}(E) \).

The following is to summarize the known facts of poset and matroid that are needed later on.

**Definition 1.** ([1, pp.1-5 & 3, pp.1-3]) (1) A poset is a set in which a binary relation \( x \leq y \) is defined, which satisfies for all \( x, y, z \) the following conditions:

(p1) \( x \leq x, \forall x \).

(p2) If \( x \leq y \) and \( y \leq x \), then \( x = y \).

(p3) If \( x \leq y \) and \( y \leq z \), then \( x \leq z \).

(2) Let \( (P, \leq) \) be a poset and \( a, b \in P \). Then \( a \) and \( b \) are comparable if \( a \leq b \) or \( b \leq a \). Otherwise, \( a \) and \( b \) are incomparable, in notation, \( a \nmid b \). A poset \( (P, \leq) \) is called a chain if it also satisfies (p4): for all \( a, b \in P \), \( a \leq b \) or \( b \leq a \).

Let \( B \neq \emptyset \) and \( B \subseteq P \). Then there is a natural partial order \( \leq_B \) on \( B \) induced by \( \leq \): for \( a, b \in B \), \( a \leq_B b \iff a \leq b \); we call \( (B, \leq_B) \) (or simply, \( (B, \leq) \)) a subposet of \( (P, \leq) \).

(3) The length, \( l(C) \), of a finite chain \( C \) is \( |C| - 1 \). A poset \( P \) is said to be of length \( n \) (in formula, \( l(P) = n \)) where \( n \) is a natural number, if and only if there is a chain in \( P \) of length \( n \) and all chains in \( P \) are of length \( \leq n \).

(4) “\( a \) covers \( b \)” in a poset \( P \) means that \( b < a \), but that \( b < x < a \) for no \( x \in P \).

(5) An isomorphism between two poset \( P \) and \( Q \) is a bijection \( \theta \) which satisfies \( x \leq y \) implies \( \theta(x) \leq \theta(y) \) and also \( \theta(x) \leq \theta(y) \) implies \( x \leq y \). Two posets \( P \) and \( Q \) are called isomorphic (in symbol, \( P \cong Q \)), if and only if there exists an isomorphism between them.
Remark 1. (1) From Definition 1, it is easy to see that in a poset \((P, \leq)\), for all \(a, b \in P\), \(a < b \iff a \leq b\) and \(a \neq b\). (2) Let \((P, \leq)\) be a poset. For convenient, sometimes, we denote it by \(P\).

Definition 2. (1) ([4, pp.7-9]) A matroid \(M\) is a finite set \(S\) and a collection \(I\) of subsets of \(S\) (called independent sets) such that (i1)-(i3) are satisfied:
- (i1) \(\emptyset \in I\).
- (i2) If \(X \in I\) and \(Y \subseteq X\), then \(Y \in I\).
- (i3) If \(U, V\) are members of \(I\) with \(|U| = |V| + 1\), there exists \(x \in U \setminus V\) such that \(V \cup x \in I\).

A base of \(M\) is a maximal element in \(I\). The collection of bases is denoted by \(\mathcal{B}(M)\).

The rank function of \(M\) is a function \(\rho : 2^S \to \mathbb{Z}\) defined by \(\rho(A) = \max(|X| : X \subseteq A, X \in I)\) \((A \subseteq S)\). The rank of the matroid, \(\rho(M)\), is the rank of the set \(S\).

Two matroids \(M_1\) and \(M_2\) on \(S_1\) and \(S_2\) respectively are isomorphic if there is a bijection \(\varphi : S_1 \to S_2\) which preserves independence. Written \(M_1 \cong M_2\) if \(M_1\) and \(M_2\) are isomorphic.

(2) ([4, p.13]) A matroid is simple if it is a matroid with no loops or parallel elements. (The definitions of loop and parallel are cf.[4, pp.12-13]).

Lemma 1. ([4, pp.8-9]) (1) It is clear that equivalently \(\varphi\) is an isomorphism between matroids \(M_1\) and \(M_2\) on \(S_1\) and \(S_2\) respectively if and only if it preserves the rank function and so on.
(2) All bases of a matroid on \(S\) have the same cardinality, which is the rank of \(S\).
(3) (Base axioms) A non-empty collection \(\mathcal{B}\) of subsets of \(S\) is the set of bases of a matroid on \(S\) if and only if it satisfies the following condition:
- (b1) If \(B_1, B_2 \in \mathcal{B}\) and \(x \in B_1 \setminus B_2\), there exists \(y \in B_2 \setminus B_1\) such that \((B_1 \cup y) \setminus x \in \mathcal{B}\).

Definition 3. (1) ([2, p.196]) A subset of \(k\) elements of \(E\) in certain contexts is called a combination of \(E\), \(k\) elements at a time, and the number of these is denoted by \(nC_k\).
(2) ([2, pp.21-22]) An equivalence relation on the set $A$ is a relation $\sim$ between $A$ and itself that has the properties:

(i) $a \sim a, \forall a \in A$;
(ii) $a \sim b \Rightarrow b \sim a$;
(iii) $a \sim b$ and $b \sim c \Rightarrow a \sim c$.

(3) ([2, p.22]) We call the splitting of a set $A$ into non-overlapping subsets a partition of $A$.

Lemma 2. ([2, p.22]) Let $\sim$ be an equivalence relation on a set $A$. Then the equivalence classes for $\sim$ constitute a partition determined equivalence relation whose equivalence classes form the partition.

Remark 2. (1) In this paper: in a poset $P$, $a$ covers $b$, in notation $b \prec a$; $b$ does not cover $a$, in notation $a \nprec b$; $(a, b) = \{x \in P : a < x \leq b\}$, $[a, b] = \{x \in P : a \leq x \leq b\}$, $(a, b) = \{x \in P : a < x < b\}$. Here, writes $C_k^n$ in place of $\binom{n}{k}$ according to my habit. (2) Based on “Base axioms” for matroids, a matroid $M$ is uniquely determined by \{A_1, \ldots, A_t\} = \mathcal{B}(M)$. Hence, we only need to find out the $\mathcal{B}(M)$ when we search $M$. We denote the cardinality of $\mathcal{B}(M)$ by $|\mathcal{B}(M)|$. (3) By [4, p.8, Theorem 2], a matroid is uniquely determined by its rank function.

2. Poset and Properties

This section firstly builds up a binary relation $\leq$ on $\mathcal{M}(E)$, and after that discusses some properties of $(\mathcal{M}(E), \leq)$.

First of all, we give the definition of a binary relation $\leq$ on $\mathcal{M}(E)$ as follows.

Definition 4. Let $M_1, M_2 \in \mathcal{M}(E)$. We distinguish four statuses to define a binary relation $\leq$ on $\mathcal{M}(E)$.

Status 1. If $M_1 = M_2$, then define $M_1 \leq M_2$.

Status 2. If $M_1 \neq M_2$ and $\rho(M_1) < \rho(M_2)$, then define $M_1 < M_2$, i.e. $M_1 \leq M_2$ is defined.

Status 3. If $M_1 \neq M_2$, $\rho(M_1) = \rho(M_2)$ and $|\mathcal{B}(M_1)| < |\mathcal{B}(M_2)|$, then define $M_1 < M_2$, i.e. $M_1 \leq M_2$ is defined.

Status 4. If $M_1 \neq M_2$, $\rho(M_1) = \rho(M_2)$ and $|\mathcal{B}(M_1)| = |\mathcal{B}(M_2)|$, then define $M_1 \nleq M_2$ and $M_2 \nleq M_1$, say, $M_1 \parallel M_2$. 
**Remark 3.** From the above definition, we see that $M_1 \leq M_2 \iff M_1 = M_2$ or $M_1 < M_2$.

The following theorem tells us that the binary relation defined above is a partial order on $\mathcal{M}(E)$.

**Theorem 1.** $(\mathcal{M}(E), \leq)$ is a poset, $I_0 = \{\emptyset\} = \min(\mathcal{M}(E), \leq)$ and $I_{|E|} = \{E\} = \max(\mathcal{M}(E), \leq)$.

**Proof.** To prove $(\mathcal{M}(E), \leq)$ is a poset, by Definition 1, we only need to check that (p1)-(p3) are satisfied by $\mathcal{M}(E)$.

Status 1 in Definition 4 implies that (p1) is correct for $\mathcal{M}(E)$.

Let $M_j \in \mathcal{M}(E)$ ($j = 1, 2$) satisfy $M_1 \leq M_2$ and $M_2 \leq M_1$. Assume $M_1 \neq M_2$.

Suppose $\rho(M_1) < \rho(M_2)$. Then by Definition 4, considering $M_1 \neq M_2$ with $M_1 \leq M_2$, it follows $M_1 < M_2$, a contradiction to $M_2 \leq M_1$. Similarly, $\rho(M_2) < \rho(M_1)$ induces a contradiction. That is to say, $\rho(M_1) = \rho(M_2)$.

Suppose $|\mathcal{B}(M_1)| < |\mathcal{B}(M_2)|$. Considering $M_1 \neq M_2$ with the above result, by status 3 in Definition 4, we obtain $M_1 < M_2$, a contradiction to $M_2 \leq M_1$. Similarly, $|\mathcal{B}(M_2)| < |\mathcal{B}(M_1)|$ induces a contradiction. That is to say, $|\mathcal{B}(M_1)| = |\mathcal{B}(M_2)|$.

However, $\rho(M_1) = \rho(M_2)$, $|\mathcal{B}(M_1)| < |\mathcal{B}(M_2)|$ and $M_1 \neq M_2$ taken with status 4 in Definition 4 together implies $M_1 \nparallel M_2$, a contradiction to $M_1 \leq M_2$ and $M_2 \leq M_1$.

Therefore $M_1 = M_2$, i.e. (p2) holds.

Let $M_j \in \mathcal{M}(E)$ ($j = 1, 2, 3$) satisfy $M_1 \leq M_2$ and $M_2 \leq M_3$. Assume $M_1 \nleq M_3$. This assumption means $M_1 \nparallel M_3$ or $M_3 < M_1$. Distinguish two cases to discuss.

**Case 1.** $M_1 \nparallel M_3$.

By Definition 4, $M_1 \nparallel M_3$ implies $M_1 \neq M_3$, $\rho(M_1) = \rho(M_3)$ and $|\mathcal{B}(M_1)| = |\mathcal{B}(M_3)|$. Besides, $M_2 \leq M_3$ and $M_1 \nparallel M_3$ together implies $M_1 \neq M_2$; $M_1 \leq M_2$ and $M_1 \nparallel M_3$ together implies $M_2 \neq M_3$.

If $\rho(M_1) < \rho(M_2)$. Then $\rho(M_3) < \rho(M_2)$. By status 2 in Definition 4, $M_3 < M_2$, a contradiction to $M_2 \leq M_3$. 


If $\rho(M_1) = \rho(M_2)$. Since $\rho(M_1) = \rho(M_3)$, one has $\rho(M_1) = \rho(M_2) = \rho(M_3)$. In light of $M_1 \leq M_2$, $M_1 \neq M_2$, $\rho(M_1) = \rho(M_2)$ and Definition 4, one has $|\mathcal{B}(M_1)| < |\mathcal{B}(M_2)|$. On the other hand, in view of $M_2 \leq M_3$, $M_2 \neq M_3$, $\rho(M_2) = \rho(M_3)$ and Definition 4, one gets $|\mathcal{B}(M_2)| < |\mathcal{B}(M_3)| = |\mathcal{B}(M_1)| < |\mathcal{B}(M_2)|$. Therefore, $|\mathcal{B}(M_2)| < |\mathcal{B}(M_3)| = |\mathcal{B}(M_1)|$, a contradiction.

Therefore, $M_1 \parallel M_3$ is wrong.

**Case 2.** $M_3 < M_1$.

$M_3 < M_1$ means “$\rho(M_3) < \rho(M_1)$” or “$\rho(M_3) = \rho(M_1)$ and $|\mathcal{B}(M_3)| < |\mathcal{B}(M_1)|$.”

When $\rho(M_3) < \rho(M_1)$. If $\rho(M_1) < \rho(M_2)$, then $\rho(M_3) < \rho(M_2)$. According to status 2 in Definition 4, it follows $M_3 < M_2$, a contradiction to $M_2 \leq M_3$ and $M_2 \neq M_3$. If $\rho(M_1) = \rho(M_2)$ and $|\mathcal{B}(M_1)| < |\mathcal{B}(M_2)|$, then $\rho(M_3) < \rho(M_2)$. According to $M_2 \neq M_3$, $\rho(M_3) < \rho(M_2)$ and Definition 4, it follows $M_3 < M_2$, a contradiction to $M_2 \leq M_3$ and $M_2 \neq M_3$. If $\rho(M_1) = \rho(M_2)$ and $|\mathcal{B}(M_1)| = |\mathcal{B}(M_2)|$, it follows a contradiction to $M_1 \leq M_2$. If $\rho(M_1) = \rho(M_2)$ and $|\mathcal{B}(M_2)| < |\mathcal{B}(M_1)|$, then it follows $M_2 < M_1$, a contradiction to $M_1 \leq M_2$.

When $\rho(M_3) = \rho(M_1)$ and $|\mathcal{B}(M_3)| < |\mathcal{B}(M_1)|$. If $\rho(M_1) < \rho(M_2)$. Then $\rho(M_3) < \rho(M_2)$, and so $M_3 \leq M_2$, a contradiction to $M_3 \neq M_2$ and $M_2 \leq M_3$. If $\rho(M_1) = \rho(M_2)$. Then $\rho(M_2) = \rho(M_3)$. Considering $M_2 \leq M_3$ with Definition 4, $|\mathcal{B}(M_2)| < |\mathcal{B}(M_3)|$ holds, and so $|\mathcal{B}(M_2)| < |\mathcal{B}(M_1)|$, further, $M_2 < M_1$, a contradiction to $M_1 \neq M_2$ and $M_1 \leq M_2$.

Therefore, $M_3 < M_1$ is wrong.

Summing up the above two cases, it follows that $M_1 \leq M_3$ is right. Say, (p3) holds.

Both $I_0 = \{\emptyset\} = \min(\mathcal{M}(E), \leq)$ and $I_{|E|} = \{E\} = \max(\mathcal{M}(E), \leq)$ are correct trivially.

Actually, it is easy to see that Definition 4 and Theorem 1 are true for all matroids defined on $E$ not only for simple matroids. Even though, from the following Property 1, we will know that in what follows, all the matroids should be simple. Next to treat the properties of $(\mathcal{M}(E), \leq)$.

**Property 1.** Let $E_j = \{a_{j1}, \ldots, a_{j|E_j|}\}$ be a finite set $(j = 1, 2)$. Then

1. $|E_1| = |E_2| \iff (\mathcal{M}(E_1), \leq) \cong (\mathcal{M}(E_2), \leq)$. 
(2) Up to isomorphism, $|E_1| < |E_2| \Leftrightarrow (\mathcal{M}(E_1), \leq)$ is isomorphic to a subposet of $(\mathcal{M}(E_2), \leq)$.

Proof. Since $|E_1| = |E_2| \Rightarrow (\mathcal{M}(E_1), \leq) \cong (\mathcal{M}(E_2), \leq)$ is straightforward, it remains to show the reverse direction. Let us finish it by three steps.

Step 1. $\{a_{j_i}\} \in \mathcal{M}(E_j)$, i.e. the set of bases of a matroid on $E_j$ has only one element $\{a_{j_i}\}$ is trivial $(i = 1, 2, \ldots, |E_j|)$, $(j = 1, 2)$. In virtue of Definition 4, it has $\mathcal{I}_0 < \{a_{j_i}\}$ $(i = 1, 2, \ldots, |E_j|)$, $(j = 1, 2)$. Suppose $\mathcal{I}_0 \not\sim \{a_{j_i}\}$ for some $i \in \{1, 2, \ldots, |E_j|\}$, i.e. $\exists X \in \mathcal{M}(E_j)$ with $\mathcal{I}_0 < X < \{a_{j_i}\}$. Since $\rho(\mathcal{I}_0) = 0$, $\rho(\{a_{j_i}\}) = 1$ and $\rho(X) \geq 0$, it has $0 \leq \rho(X) \leq 1$. If $\rho(X) = 0$, then it must have $X = \mathcal{I}_0$, a contradiction. Thus $\rho(X) = 1$. Then the set of bases of $X$ has at least an element which is not $0$, i.e. $0 < |\mathcal{B}(X)|$. On the other hand, $X < \{a_{j_i}\}$ hints $|\mathcal{B}(X)| < |\mathcal{B}(\{a_{j_i}\})| = 1$. That is, $0 < |\mathcal{B}(X)| < 1$, a contradiction. Hence $\mathcal{I}_0 < \{a_{j_i}\} (i = 1, 2, \ldots, |E_j|)$ $(j = 1, 2)$.

Step 2. Suppose $Y \in \mathcal{M}(E_j)$ covers $\mathcal{I}_0$ in $\mathcal{M}(E_j)$ but not be an element in $\{a_{i_j}\} : i = 1, 2, \ldots, |E_j|\}$, $(j \in \{1, 2\})$. $\mathcal{I}_0 < Y$ tells us the holding of “$0 = \rho(\mathcal{I}_0) < \rho(Y)$” or “$0 = \rho(\mathcal{I}_0) = \rho(Y)$ but $1 = |\mathcal{B}(\mathcal{I}_0)| < |\mathcal{B}(Y)|$”. The latter state is wrong trivially. The former state suggests $1 \leq \rho(Y)$. If $1 < \rho(Y)$, then there exists $a_{j_i} \in E_j$ such that $\{a_{j_i}\} < Y$ which is a contradiction with $\mathcal{I}_0 < Y$. Hence $\rho(Y) = 1$, say, $Y = \{A_1, \ldots, A_t\}$ where $A_p \subseteq E_j$ and $|A_p| = 1$ $(p = 1, 2, \ldots, t)$. At this time, $1 < t$ gives $\{a_{j_i}\} < Y$, $\forall i \in \{1, 2, \ldots, |E_j|\}$, and so, $\mathcal{I}_0 < Y$ is not true. Therefore, $t = 1$. This leads us to a conclusion that there is an $\{a_{j_i}\}$ such that $Y = \{a_{j_i}\}$, a contradiction.

Namely, in $\mathcal{M}(E_j)$, by Step 1 and Step 2, the set of atoms of $(\mathcal{M}(E_j), \leq)$, say, all the elements in $\mathcal{M}(E_j)$ that cover $\mathcal{I}_0$, is $\{\{a_{j_i}\} : i = 1, 2, \ldots, |E_j|\}$, $(j = 1, 2)$.

Step 3. $(\mathcal{M}(E_1), \leq) \cong (\mathcal{M}(E_2), \leq)$ means that there is an isomorphic map $\varphi$ between $(\mathcal{M}(E_1), \leq)$ and $(\mathcal{M}(E_2), \leq)$. In virtue of Definition 1 and Definition 2, $\varphi(\mathcal{I}_0) = \mathcal{I}_0$, and $\varphi$ is a bijection between $\{a_{1_i}\} : i = 1, 2, \ldots, |E_1|\}$ and $\{a_{2_i} : i = 1, 2, \ldots, |E_2|\}$. Hence $|E_1| = |E_2|$.

The result of (2) is straightforward from the above (1).

Remark 4. I. Connell says in [2, p.70] that if “$A$ and $B$ are isomorphic”, then “$A$ and $B$ are ‘abstractly the same’ or ‘copies of one another’”. Hence,
Property 1 makes the topic and results in this paper be significant and suit for general, not only for a special given set.

**Property 2.** (1) \( \{\mathcal{I}_s, \mathcal{I}_{s+1} : s = 0, 1, \ldots, |E| - 1\} \cup \{\mathcal{I}_0\} \) is a partition of \( \mathcal{M}(E) \), where \( \mathcal{I}_g = \{B_{gj} : B_{gj} \subseteq E, |B_{gj}| = g, j = 1, 2, \ldots, C_{|E|}^g\}, (g = 1, 2, \ldots, |E|). 

(2) The atoms of \( ([\mathcal{I}_s, \mathcal{I}_{s+1}], \leq) \) are \( \{\{B_j\} : j = 1, 2, \ldots, C_{|E|}^{s+1}\} \) where \( B_j \subseteq E \) and \(|B_j| = s + 1, (j = 1, 2, \ldots, C_{|E|}^{s+1} ; s = 0, 1, \ldots, |E| - 1\).

**Proof.** (1) Theorem 1 and Definition 4 taken together tells us that \( \mathcal{I}_0 \) and \( \mathcal{I}_{|E|} \) satisfy \( \mathcal{I}_0 < M < \mathcal{I}_{|E|} \) for \( \forall M \in \mathcal{M}(E) \backslash \{\mathcal{I}_0, \mathcal{I}_{|E|}\} \). Since \( (\mathcal{I}_0, \mathcal{I}_0) = \mathcal{I}_0 \notin \{\{\mathcal{I}_s, \mathcal{I}_{s+1} : s = 0, 1, \ldots, |E| - 1\} \) and \( \mathcal{I}_{|E|-1}, \mathcal{I}_{|E|} = \{E\} \), it remains only to prove that any \( M \in \mathcal{M}(E) \) which \( 0 < \rho(M) < |E| \) belongs to one and only one \((\mathcal{I}_s, \mathcal{I}_{s+1}) \) \((0 \leq s \leq |E| - 1)\).

We know that \( \rho(M) \) is a fixed number for \( \forall M \in \mathcal{M}(E) \backslash \{\mathcal{I}_0, \mathcal{I}_{|E|}\} \), that is, there is one and only one \( s \in \{0, 1, 2, \ldots, |E| - 1\} \) satisfying \( \rho(M) = s \). Herein, there is only one \( \mathcal{I}_s \) for some \( s \in \{0, 1, 2, \ldots, |E| - 1\} \) satisfying \( M \subseteq \mathcal{I}_s \). In addition, \( \rho(\mathcal{I}_{s-1}) = s - 1 \) taken with \( \rho(\mathcal{I}_s) = s \) and \( M \subseteq \mathcal{I}_s \) together hints that \( \mathcal{I}_{s-1} < M \leq \mathcal{I}_s \), i.e. \( M \in (\mathcal{I}_{s-1}, \mathcal{I}_s) \).

Adding up, we obtain that \( \{\{\mathcal{I}_s, \mathcal{I}_{s+1} : s = 0, 1, \ldots, |E| - 1\} \cup \{\mathcal{I}_0\} \) is a partition of \( \mathcal{M}(E) \).

(2) By the proof of (1), if the set of bases of a matroid on \( E \) is made up by only one element \{\( B_j \)\}, then \( \{B_j\} \in (\mathcal{I}_s, \mathcal{I}_{s+1}) \) \( (j = 1, 2, \ldots, C_{|E|}^{s+1}) \). Since \( \rho(\mathcal{I}_s) = s \) and \( \rho(B_j) = s + 1 \) imply \( \mathcal{I}_s < \{B_j\} \) for \( j = 1, 2, \ldots, C_{|E|}^{s+1} \). Analogous to Step 1 and Step 2 in the proof of Property 1, the needed follows.

**Corollary 1.** For any \( M \in \mathcal{M}(E) \backslash \mathcal{I}_0 \), \( M \in (\mathcal{I}_{s-1}, \mathcal{I}_s) \) if and only if \( \rho(M) = s, (1 \leq s \leq |E|)\).

**Proof.** It is straightforward from Property 2 and \( \rho(M) \) that is a character of \( M \).

**Remark 5.** (1) Let \( M \in \mathcal{M}(E) \) where \( \mathcal{B}(M) = \{A_1, A_2, \ldots, A_t\} \) \((1 < t \leq C_{|E|}^s, \rho(M) = s \) and \( A_j = \{a_1, a_2, \ldots, a_{s-1}, a_{js}\} \) \((j = 1, 2, \ldots, t)\). Evidently \( \{A_1, A_2, \ldots, A_{t-1}\} \) is the set of bases of a matroid \( M_{t-1} \in \mathcal{M}(E) \) with \( \rho(M_{t-1}) = \).
\[ \rho(M) \text{ and } |B(M_{t-1})| = |B(M)| - 1. \] In addition \( M_{t-1} \prec M \). Therefore, \( \mathcal{I}_{s-1} \prec M_1 \prec M_2 \prec \ldots \prec M_{t-1} \prec M \), where \( B(M_1) = \{A_1\}, B(M_2) = \{A_1, A_2\}, \ldots , B(M_{t-1}) = \{A_1, A_2, \ldots , A_{t-1}\} \).

(2) We assert that for any \( A \subseteq E \), \( |A| = s \) and \( 1 \leq t \leq C_{|E|}^s \), there exists \( M \in \mathcal{M}(E) \) with \( |B(M)| = t \) and \( \rho(M) = s \) satisfying \( A \in B(M) \). Actually, put \( B = \{A = \{a_1, a_2, \ldots , a_s-1, a_{1s}\}, A_2 = \{a_1, a_2, \ldots , a_s-1, a_{2s}\}, \ldots , A_t = \{a_1, a_2, \ldots , a_{s-1}, a_{ts}\}\}. \) Then according to the above (1), \( B \) is the set of bases of a matroid \( M \) on \( E \), this is, \( M \in \mathcal{M}(E) \) and \( B(M) = B \).

Furthermore, it is easier to see that for any \( M_1, M_2 \in \mathcal{M}(E) \), \( M_1, M_2 \in (\mathcal{I}_{s-1}, \mathcal{I}_s), B(M_1) \subseteq B(M), M_1 \prec M \), then there exists a maximal \( M_2 \in \mathcal{M}(E) \) satisfying \( B(M) \setminus B(M_1) \subseteq B(M_2) \) and \( B(M_1) \cup (B(M) \cap B(M_2)) = B(M_1) \cup (B(M) \setminus B(M_2)) = B(M) \). Actually, we have \( |B(M) \setminus B(M_1)| = 1 \) and denote \( A = B(M) \setminus B(M_1) \), and so \( \{A\} \in \mathcal{M}(E), B(\{A\}) = 1, B(M) = B(M_1) \cup B(\{A\}) \).

It is not difficult to get a maximal \( M_2 \) satisfying \( A \in B(M_2), B(M_2) \setminus A \subseteq B(M_1) \) and \( B(M_1) \cup (B(M) \cap B(M_2)) = B(M_1) \cup (B(M) \setminus B(M_2)) = B(M) \).

**Property 3.** Let \( M_1, M_2 \in \mathcal{M}(E) \) and \( M_1 \neq M_2 \). Then, in \((\mathcal{M}(E), \leq)\), \( M_1 \prec M_2 \) holds if and only if one of the following cases is happened.

**Case 1.** \( \exists s \in \{0, 1, \ldots , |E| - 1\} \) satisfies \( M_1 = \mathcal{I}_s \). Meanwhile, \( M_2 \) is an atom of \([\mathcal{I}_s, \mathcal{I}_{s+1}]\).

**Case 2.** \( \rho(M_1) = \rho(M_2) \) and \( |B(M_1)| = |B(M_2)| - 1 \).

**Proof.** \((\Leftarrow\Rightarrow)\) Routine verification.

\((\Rightarrow)\) Let \( M_1 \prec M_2 \). It should notice that by Definition 4 and \( M_1 \neq M_2, M_1 \prec M_2 \) implies that status 2 or status 3 in Definition 4 will happen. Besides, \( M_1 \prec M_2 \) tells us \( M_1 \neq \mathcal{I}_{|E|} \). We divide into two steps to prove.

Step 1. Status 2 happens. This induces \( \rho(M_1) < \rho(M_2) \).

If Case 1 is not true, i.e., no \( \mathcal{I}_s \) suits for \( M_1 = \mathcal{I}_s, (\forall s \in \{0, 1, \ldots , |E| - 1\}) \). We notice that there exists an \( s \in \{0, 1, \ldots , |E| - 1\} \) satisfying \( \rho(M_1) = s + 1 \), and so \( M_1 \in (\mathcal{I}_s, \mathcal{I}_{s+1}] \) by Corollary 1. On the other hand, \( s + 1 = \rho(M_1) < \rho(M_2) = k \), and further \( s + 2 \leq \rho(M_2) = k \). This indicates that \( M_2 \in (\mathcal{I}_{k-1}, \mathcal{I}_k) \neq (\mathcal{I}_s, \mathcal{I}_{s+1}] \).

Suppose \( \mathcal{I}_{s+2} = \mathcal{I}_{|E|} \). Then \( M_2 = \mathcal{I}_{|E|} \). Therefore \( M_1 = \mathcal{I}_{|E|-1} \), contradiction with the supposition of “Case 1 is not true”. This gives \( \mathcal{I}_{s+2} < \mathcal{I}_{|E|} \). Since
$M_1 \neq I_{s+1}, M_1 \leq I_{s+1}, I_{s+2} < I_{|E|}, \rho(I_{s+1}) = s + 1 < s + 2 \leq \rho(M_2)$ taken with Property 2 and the supposition together follows $M_1 < I_{s+1} \prec \{B_{(s+2)_j}\} \leq I_{s+2} \preceq M_2$, where $B_{(s+2)_j} \subseteq E, |B_{(s+2)_j}| = s + 2, (j = 1, 2, \ldots, C_{|E|}^{s+2})$, and so $M_1 \neq M_2$.

In other words, when status 2 happens, it must have Case 1 to be true.

Step 2. Status 3 happens. This induces $\rho(M_1) = \rho(M_2)$ and $|B(M_1)| < |B(M_2)|$, and further, $|B(M_1)| + 1 \leq |B(M_2)|$. By Remark 5, it has $B(M_3) = \{A_1, A_2, \ldots, A_t\}$ where $t = |B(M_2)| - 1, A_j = (a_1, a_2, \ldots, a_{s-1}, a_{1s}), (j = 1, 2, \ldots, t)$ is the set of bases of a matroid $M_3 \in \mathcal{M}(E)$ and so by Definition 4, $M_3 \prec M_2$.

In addition, Definition 4 shows us that $M_1 \prec M_2$ and $M_3 \prec M_2$ should induce $M_1 || M_3$ and hence Case 2 holds.

That is to say, when status 3 happens, Case 2 should be true.

**Corollary 2.** (1) Let $M \in \mathcal{M}(E) \setminus I_0$ and $B(M) = \{A_1, A_2, \ldots, A_t\}$ ($1 < t$). Then there exists an $M_1 \in \mathcal{M}(E)$ satisfying $M_1 \prec M$.

(2) Any maximal chain in $[I_s, M]$ and in $[I_s, I_{s+1}]$ has the same length $l([I_s, M]) = t$ and $l([I_s, I_{s+1}]) = C_{|E|}^{s+1}$ respectively, where $\rho(M) = s + 1$ and $|B(M)| = t, (s = 0, 1, \ldots, |E| - 1)$. The length of $[I_0, I_0]$ is 0.

(3) In $(\mathcal{M}(E), \leq)$, any maximal chain has the same length $l(\mathcal{M}(E)) = 2^{|E|} - 1$.

(4) Let $M_1, M_2 \in \mathcal{M}(E)$ and $M_1 \neq M_2$. Then $M_1 || M_2 \iff l([I_0, M_1]) = l([I_0, M_2]) \iff \exists s$ satisfies $l([I_s, M_1]) = l([I_s, M_2])$ where $(1 \leq s \leq |E| - 1)$.

**Proof.** (1) If $M = I_{|E|}$, then $M_1 = I_{|E| - 1} \prec I_{|E|}$ is obvious. If $M \neq I_{|E|}, \rho(M) = s + 1$ and $t = 1$, then by Property 2, $M$ is an atom of $[I_s, I_{s+1}]$, and so $M_1 = I_s \prec M$. Since $(\mathcal{M}(E), \leq)$ is a poset, by Remark 5, there exists $M_1 \in \mathcal{M}(E)$ such that $B(M_1) = \{A_1 = (a_1, a_2, \ldots, a_{s-1}, a_{1s}), A_2 = (a_1, a_2, \ldots, a_{s-1}, a_{2s}), \ldots, A_{t-1} = (a_1, a_2, \ldots, a_{s-1}, a_{(t-1)s})\}$. Evidently, $M_1 \prec M$.

(2) By Remark 5, Property 3 and the above (1), one gets that any maximal chain in $[I_s, M]$ and $[I_s, I_{s+1}]$ has length $t$ and $C_{|E|}^{s+1}$ respectively. Namely, $l([I_s, M]) = |B(M)|$ and $l([I_s, I_{s+1}]) = C_{|E|}^{s+1}$.

$l([I_0, I_0]) = 0$ is trivial.

(3) By Property 2 and the above (1) and (2), one has $l(\mathcal{M}(E)) = C_{|E|}^1 + C_{|E|}^2 + \ldots + C_{|E|}^{|E|} = 2^{|E|} - 1$. 


(4) It is straightforward from Definition 4, Corollary 1, Remark 5 and Property 3.

**Property 4.** Let \( M_j \in \mathcal{M}(E) \) and \(|\mathcal{B}(M_1)| = t, \rho(M_1) = s \). If \( M_1 \parallel M_2 \) and \( M_1, M_2 \) satisfy the following (i) and one of cases in (ii), then \( \mathcal{B}(M_1) \cup \mathcal{B}(M_2) \) is the set of bases of a matroid \( M_1 \cup M_2 \in \mathcal{M}(E) \).

(i) \(|\mathcal{B}(M_1) \cap \mathcal{B}(M_2)| = |\mathcal{B}(M_1)| - 1 = t - 1 \).

(ii) For all \( A_{11} \in \mathcal{B}(M_1) \setminus \mathcal{B}(M_2), A_{22} \in \mathcal{B}(M_2) \setminus \mathcal{B}(M_1) \), it has one of the following cases happens.

**Case 1.** \(|A_{11} \cap A_{22}| = s - 1 \).

**Case 2.** \(|A_{11} \cap A_{22}| < s - 1 \) and for \( \forall a \in A_p \setminus A_q, \exists y \in A_q \setminus A_p \) satisfies \((A_p \setminus a) \cup y \in \mathcal{B}(M_1) \cap \mathcal{B}(M_2), \) where \((p \neq q; p, q \in \{11, 22\})\).

**Proof.** \( M_1 \parallel M_2 \) hints by Definition 4 that \( \rho(M_1) = \rho(M_2) = s \) \((0 < s < |E|)\) and \(|\mathcal{B}(M_1)| = |\mathcal{B}(M_2)| = t - 1 \) \((2 \leq t \leq C_\text{r}^s\}_E)\). Furthermore, \( M_j \in (\mathcal{I}_{s-1}, \mathcal{I}_s) \setminus \mathcal{I}_s \ (j = 1, 2). \) Since \(|\mathcal{B}(M_1) \cap \mathcal{B}(M_2)| = |\mathcal{B}(M_1)| - 1 = t - 2 \), it follows \(|\mathcal{B}(M_1) \setminus \mathcal{B}(M_2)| = |\mathcal{B}(M_1)| - |\mathcal{B}(M_1) \cap \mathcal{B}(M_2)| = (t - 1) - (t - 2) = 1\), similar, \(|\mathcal{B}(M_2) \setminus \mathcal{B}(M_1)| = 1\), and so \( \mathcal{B}(M_1) = \{A_1, A_3, \ldots, A_{t-2}, A_{t-1}\}, \mathcal{B}(M_2) = \{A_2, A_3, \ldots, A_{t-2}, A_{t-1}\}\), and hence, \( \mathcal{B}(M_1) \setminus \mathcal{B}(M_2) = A_1, \mathcal{B}(M_2) \setminus \mathcal{B}(M_1) = A_2. \)

By Lemma 1, the asking for \( M_1 \cup M_2 \in \mathcal{M}(E) \) where \( \mathcal{B}(M_1 \cup M_2) = \mathcal{B}(M_1) \cup \mathcal{B}(M_2) \) remains only to prove the next question:

What is the prerequisite for the truth of the following: for \( \forall a \in A_p \setminus A_q, \exists y \in A_q \setminus A_p \) satisfying \((A_p \setminus a) \cup y \in \mathcal{B}(M_1 \cup M_2), \) \((p \neq q; p, q \in \{1, 2\})\)\

Dividing into three steps to answer this question.

**Step 1.** If \((A_p \setminus a) \cup y = A_p\), then \( y \in A_q \setminus A_p\).

**Step 2.** If \((A_p \setminus a) \cup y = A_q\), then \( A_p \setminus a \subseteq A_q\), and so \(|A_p \cap A_q| = |A_p \setminus a| = s - 1\), i.e. \( A_p = (a, a_1, \ldots, a_{s-1}), A_q = (y, a_1, \ldots, a_{s-1})\). Hence the Case 1 in (ii) happens.

**Step 3.** If \((A_p \setminus a) \cup y \notin \{A_p, A_q\}\), then \((A_p \setminus a) \cup y \in \mathcal{B}(M_1) \cap \mathcal{B}(M_2)\) holds.

This just gives another answer, this is, the Case 2 in (ii) should happen.

Summarizing the above steps, we have that:

**Case 1.** \(|A_1 \cap A_2| = s - 1.\)
Case 2. \(|A_1 \cap A_2| < s - 1\) and for \(\forall a \in A_p \setminus A_q, \exists y \in A_q \setminus A_p\) satisfies \((A_p \setminus a) \cup y \in \mathcal{B}(M_1) \cap \mathcal{B}(M_2)\), where \((p \neq q; p, q \in \{1, 2\})\).

No matter which case happens, it has \(M_1 \cup M_2 \in \mathcal{M}(E)\) with the set of its bases as \(\mathcal{B}(M_1) \cup \mathcal{B}(M_2)\). \(M_j \prec M_1 \cup M_2\) is an evident fact \((j = 1, 2)\).

Corollary 3. (1) Let \(M, M_1, M_2 \in \mathcal{M}(E), \rho(M) = s, \mathcal{B}(M) = \mathcal{B}(M_1) \cup \mathcal{B}(M_2), \mathcal{B}(M_1) \cap M_2 = t, \) where \(1 \leq t < C_s^2\). Then \(M_1, M_2 \prec M \iff (i)\) holds and one of cases in (ii) happens, where both (i) and (ii) are described as in Property 4.

(2) Let \(M \in \mathcal{M}(E), \rho(M) = s\) and \(|\mathcal{B}(M)| = 2\). Then it must have two members \(M_1, M_2 \in \mathcal{M}(E), M_1, M_2 \prec M\) satisfies \(M_1||M_2\) and both (i) and Case 1 in (ii) hold. Conversely, if there exist \(M_1, M_2 \in \mathcal{M}(E)\) such that \(M_1||M_2\) and both (i) and Case 1 in (ii), then \(\mathcal{B}(M_1) \cup \mathcal{B}(M_2)\) is the set of a matroid on \(E\) and \(M_j \prec M\) \((j = 1, 2)\).

Proof. (1) \((\iff)\) Routine verification from Property 4 and Definition 4.

\((\Rightarrow)\) \(M_1, M_2 \prec M, M_1||M_2\) Corollary 2, Definition 4 and Property 3 together tells us \(\rho(M_1) = \rho(M_2) = s\) and \(|\mathcal{B}(M_1)| = |\mathcal{B}(M_2)| = |\mathcal{B}(M)| - 1\), i.e. \(|\mathcal{B}(M)| = t + 1\). Since \(\mathcal{B}(M) = \mathcal{B}(M_1) \cup \mathcal{B}(M_2)\) implies \(\mathcal{B}(M_1), \mathcal{B}(M_2) \subseteq \mathcal{B}(M)\).

Suppose (i) does not hold, say \(|\mathcal{B}(M_1) \cap \mathcal{B}(M_2)| < t - 1\). Then in light of \(M_1||M_2\) and Definition 4, \(|\mathcal{B}(M_1) \setminus (\mathcal{B}(M_1) \cap \mathcal{B}(M_2))| = |\mathcal{B}(M_2) \setminus (\mathcal{B}(M_1) \cap \mathcal{B}(M_2))| > t - t = |\mathcal{B}(M)| > |\mathcal{B}(M_1)||M_2| = |\mathcal{B}(M)| + |\mathcal{B}(M_1)||M_2| > t + 1 = |\mathcal{B}(M)|\), a contradiction, and hence (i) holds. Say, \(\mathcal{B}(M_1) = (A_1, \ldots, A_{t-1}, A_{1t}), \mathcal{B}(M_2) = (A_1, \ldots, A_{t-1}, A_{2t})\).

If Case 1 in (ii) does not hold, this means \(|A_{1t} \cap A_{2t}| < s - 1\). Considering with Lemma 1 and \(M \in \mathcal{M}(E)\), it follows that Case 2 in (ii) should happen.

If Case 2 in (ii) does not hold, say, \(\exists a \in A_p \setminus A_q, \forall y \in A_q \setminus A_p, \) it has \((A_p \setminus a) \cup y \notin \mathcal{B}(M_1) \cap \mathcal{B}(M_2)\), where \((p \neq q; p, q \in \{1t, 2t\})\). No matter to suppose \(a \in A_{1t} \setminus A_{2t}\). Since Lemma 1 and \(M \in \mathcal{M}(E)\), there exists \(y \in A_{2t} \setminus A_{1t}\) satisfying \((A_{1t} \setminus a) \cup y \in \mathcal{B}(M)\), and so \((A_{1t} \setminus a) \cup y = A_2\), and hence \(|A_{1t} \cap A_{2t}| = |A_{1t}| - 1 = \rho(M) - 1\), this means that Case 1 in (ii) happens.

(2) By Property 2, Property 4 and \(|\mathcal{B}(M)| = t = 2\), it is easy to get the need.

Remark 6. (1) From the proof of (2) in Corollary 3, we get that for \(\forall M \in \mathcal{M}(E), \rho(M) = s\) and \(|\mathcal{B}(M)| = 2\), it exists \(M_1, M_2 \in \mathcal{M}(E), M_j \prec M\)
and \( \mathcal{B}(M) = \mathcal{B}(M_1) \cup \mathcal{B}(M_2) \). Actually, the (2) in Corollary 3 gives a constructive approach to obtain the whole elements in \( \mathcal{M}_{s_2} = \{ M \in \mathcal{M}(E) \mid \rho(M) = s, |\mathcal{B}(M)| = 2 \} \).

(2) Let \( M \in \mathcal{M}(E) \), \( \rho(M) = s > 1 \) and \( 1 < |\mathcal{B}(M)| = t < C|_{E|}^s \). Let \( \mathcal{M}_{s(t-1)} = \{ M' \mid M' \in \mathcal{M}(E), M' \prec M \}, \mathcal{M}_{s(t-2)} = \{ M'' \mid M'' \in \mathcal{M}(E), \exists M' \in \mathcal{M}_{s(t-1)} \text{ satisfies } M'' \prec M' \}, \ldots, \mathcal{M}_{s1} = \{ M^{(1)} \mid M^{(1)} \in \mathcal{M}(E), \exists M^{(2)} \in \mathcal{M}_{s2} \text{ satisfying } M^{(1)} \prec M^{(2)} \} \). It is easy to see that \( \mathcal{M}_{s1} \) is the set of atoms in \([\mathcal{I}_{s-1}, \mathcal{I}_s]\). By Property 2 and Corollary 2, we only need to find a way to search all the members in \([\mathcal{I}_{s-1}, \mathcal{I}_s]\) for any \( s, (1 \leq s \leq |E|) \). By the definition of \( \mathcal{I}_{s-1}, \mathcal{I}_s \), one gets \( \mathcal{I}_{s-1} \) and \( \mathcal{I}_s \). Considering the above (1) and Property 2, now, we need to search matroids in \([\mathcal{I}_{s-1}, \mathcal{I}_s]\) for \( |\mathcal{B}(M)| = t, 3 \leq t < C|_{E|}^s \). Actually, for any \( M' \in \mathcal{M}(E) \), \( \rho(M') = s \) and \( |\mathcal{B}(M')| = |\mathcal{B}(M)| - 1 = t - 1 \), it must have \( M' \prec M \), this is, \( M' \in \mathcal{M}_{s(t-1)} \). We can obtain some of the members in \( \mathcal{M}_{st} \) by Property 4. The following is to give a way to find the others in \( \mathcal{M}_{st} \).

Suppose \( M_1 \in \mathcal{M}_{s(t-1)}, \mathcal{B}(M_1) \subset \mathcal{B}(M) \), and for any \( M' \in \mathcal{M}_{s(t-1)}, \mathcal{B}(M_1) \cup \mathcal{B}(M') \neq \mathcal{B}(M) \). Let \( A \in \mathcal{B}(M) \setminus \mathcal{B}(M_1) \). Then by Remark 5, we know that there exists \( M_2 \in \mathcal{M}_{s(t-1)} \) satisfying \( A \in \mathcal{B}(M_2) \setminus \mathcal{B}(M_1) \) and \( \mathcal{B}(M_1) \cup \{ A \} = \mathcal{B}(M) \). That is to say, we only need to check \( \forall a \in A \setminus A_p, \) there exists or not \( y \in A_p \setminus A \) satisfying \( (A \setminus a) \cup y \in \mathcal{B}(M_1) \), and simultaneously, \( \forall a \in A_p \setminus A, \) there exists or not \( y \in A_p \setminus A \) satisfying \( (A_p \setminus a) \cup y \in \mathcal{B}(M_1) \cup \{ A \} \) for any \( A_p \in \mathcal{B}(M_1) \). If \( y \) in the above both cases are existed, then \( \mathcal{B}(M_1) \cup \{ A \} \) is the set of bases of \( M \); otherwise, \( \mathcal{B}(M_1) \cup \{ A \} \) will not be the set of bases of a matroid on \( E \).

Up to now, we get all the members of \( \mathcal{M}_{st} \) relative with \( \mathcal{M}_{s(t-1)} \) denoted by \( \mathcal{T}_{st}(^{t-1}) \). We see that all these members obtained by the above and Property 4 has the following property: \( \forall M \in \mathcal{T}_{st}(^{t-1}), \) there is \( M_1 \in \mathcal{M}_{s(t-1)} \) satisfying \( \mathcal{B}(M_1) \subset \mathcal{B}(M) \). We call the members in \( \mathcal{T}_{st}(^{t-1}) \) produced by \( \mathcal{M}_{s(t-1)} \). By Definition 2, it easily follows that the matroids in \( \mathcal{T}_{st}(^{t-1}) \) obtained from Property 4 are non-isomorphic to that obtained by the above in \( \mathcal{T}_{st}(^{t-1}) \).

Next to answer how to get the others in \( \mathcal{M}_{st} \), i.e. produced by \( \mathcal{M}_{s(t-2)}, \ldots, \) and \( \mathcal{M}_{s1} \).

Let \( M_1 \in \mathcal{M}_{s(t-2)} \) and for any \( M_2 \in \mathcal{M}_{s(t-2)}, \) \(|\mathcal{B}(M_1) \cup \mathcal{B}(M_2)| \neq t - 1 \), say, \(|\mathcal{B}(M_1) \cup \mathcal{B}(M_2)| > t - 1 \). We just want to know that under what conditions, \( \mathcal{B}(M_1) \cup \mathcal{B}(M_2) \) is the set of bases of a matroid \( M \in \mathcal{M}(E) \) with \( \rho(M) = s \) and \( |\mathcal{B}(M)| = t \). Divided two steps to finish the discussion.
Step 1. Suppose $\mathcal{B}(M_1) \cup \mathcal{B}(M_2) = \mathcal{B}(M)$ for an $M \in \mathcal{M}(E)$ with $\rho(M) = s$ and $|\mathcal{B}(M)| = t$, where $M_2 \in \mathcal{M}_{s(t-1)}$. Then

(i) $|\mathcal{B}(M_1) \cap \mathcal{B}(M_2)| = |\mathcal{B}(M_1)| - 2$.

This is induced from that $\mathcal{B}(M_1) \neq \mathcal{B}(M_2)$ and $|\mathcal{B}(M_1) \cup \mathcal{B}(M_2)| = t$. Hence we can assume $\mathcal{B}(M_j) = \{A_1, \ldots, A_{t-2}, A_1(t-1), A_{1t}\}$ $(j = 1, 2)$.

(ii) For any $a \in A_{jp} \setminus A_{iq}$, $\exists y \in A_{iq} \setminus A_{jp}$ satisfying $(A_{jp} \setminus a) \cup y \in \mathcal{B}(M_1) \cup \mathcal{B}(M_2)$, $(i, j = 1, 2; i \neq j; p, q = t - 1, t; p \neq q)$.

This is induced by bases axioms.

Conversely, when (i) and (ii) hold, we see that $\mathcal{B}(M_1) \cup \mathcal{B}(M_2) = \mathcal{B}(M)$ for some $M \in \mathcal{M}(E)$ with $\rho(M) = s$ and $|\mathcal{B}(M)| = t$.

Step 2. Suppose $\forall M_2 \in \mathcal{M}(E)$, if $M_1 \parallel M_2$, then $|\mathcal{B}(M_1) \cup \mathcal{B}(M_2)| \neq t$.

We want to know that under what conditions $\mathcal{B}(M_1) \cup \mathcal{B}(M_2)$ will be true for some $M \in \mathcal{M}(E), \rho(M) = s$ for $A_1, A_2 \in \mathcal{M}_{s1}, A_1 \neq A_2$ and $A_1, A_2 \notin \mathcal{B}(M_1)$.

If $A_1, A_2$ belong to the same $\mathcal{B}(M_2)$ for some $M_2 \in \mathcal{M}_{s(t-2)}$, then by bases axioms,

(iii) $|A_j| = s, A_j \in \mathcal{B}(M_2) \setminus \mathcal{B}(M_1)$ $(j = 1, 2)$.

(iv) For $\forall a \in A_j \setminus B_i$, $\exists y \in B_i \setminus A_j$ satisfies $(A_j \setminus a) \cup y \in \mathcal{B}(M_1) \cup \{A_1, A_2\}$, and at the same time, for $\forall a \in B_i \setminus A_j$, $\exists y \in A_j \setminus B_i$ satisfies $(B_i \setminus a) \cup y \in \mathcal{B}(M_1) \cup \{A_1, A_2\}$, where $j = 1, 2; B_i \in \mathcal{B}(M_1), (i = 1, 2, \ldots, |\mathcal{B}(M_1)|)$.

Analogously, by base axioms, if (iii) and (iv) hold, then $\mathcal{B}(M_1) \cup \{A_1, A_2\}$ is the bases of a matroid in $\mathcal{M}_{st}$.

If $A_1, A_2$ does not belong to the same $\mathcal{B}(M_2)$ where $M_4 \in \mathcal{M}_{s(t-2)}$. Suppose $A_1 \in \mathcal{B}(M_2) \setminus \mathcal{B}(M_1)$ and $A_2 \notin \mathcal{B}(M_2) \setminus \mathcal{B}(M_1)$ for some $M_2 \in \mathcal{M}_{s(t-2)}$. If $\forall a \in A_1 \setminus B_i$, $\exists y \in B_i \setminus A_1$ satisfies $(A_1 \setminus a) \cup y \in \mathcal{B}(M_1) \cup \{A_1\}$ or $(A_1 \setminus a) \cup y \in \{A_2\}$, and at the same time, $\forall a \in B_i \setminus A_1$, $\exists y \in A_1 \setminus B_i$ satisfies $(B_i \setminus a) \cup y \in \mathcal{B}(M_1) \cup \{A_1\}$ or $(B_i \setminus a) \cup y \in \{A_2\}$, where $B_i \in \mathcal{B}(M_1), B_i \subseteq E$ and $|B_i| = s, (i = 1, 2, \ldots, |\mathcal{B}(M_1)|)$. Simultaneously, $\forall a \in A_2 \setminus D$, $\exists y \in D \setminus A_2$ satisfies $(A_2 \setminus a) \cup y \in \mathcal{B}(M_1) \cup \{A_1\}$ where $D \in \mathcal{B}(M_1) \cup \{A_1\}$. Then $\mathcal{B}(M_1) \cup \{A_1\} \cup \{A_2\} = \mathcal{B}(M)$ for some $M \in \mathcal{M}(E), \rho(M) = s$ and $|\mathcal{B}(M)| = t$. Of course, since by Remark 5, $\exists \mathcal{B}(M_3) \in \mathcal{M}_{s(t-2)}$ satisfies $A_2 \in \mathcal{B}(M_3)$ and $M_i \parallel M_j, (i \neq j; i, j = 1, 2, 3)$. Hence, we only need to consider the elements in such a $M_3 \in \mathcal{M}_{s(t-2)}$ satisfying $A_2 \in \mathcal{B}(M_3) \setminus \mathcal{B}(M_1)$ and $A_2 \notin \mathcal{B}(M_2) \setminus \mathcal{B}(M_1)$. 

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By the two steps, we obtained all the members in $M_{st}$ produced by $M_{s(t-2)}$; written $T_{st}^{(t-2)}$. In induction way, finally, we will obtain all the members in $M_{st}$ produced by $M_{s(t-3)}, M_{s(t-4)}, \ldots$, and $M_{s1}$. In virtue of the process above, $M \in M_{st}$ is produced by at least one $M_{sj} (1 \leq j \leq t-1)$.

(3) For searching $M \in \mathcal{M}(E)$ with $\rho(M) = s$ and $|\mathcal{B}(M)| = t$, we see that when $t - k = t - (t - k)$ for $1 \leq k \leq t - 1$, according to the method introduced above, $T_{st}^{(t-k)} = T_{st}^{(t-(t-k))}$ hold. Hence when $2k = t$, it only needs to consider one between $T_{st}^{(t-k)}$ and $T_{st}^{(t-(t-k))}$. Besides, when $t - i \neq t - j (1 \leq t - i, t - j \leq t - 1)$ by Definition 2, every element in $T_{st}^{(t-k)}$ is non-isomorphic to a member in $T_{st}^{(t-j)}$, where $t - k \neq t - j; t - k \neq t - (t - j); 1 \leq t - k, t - j \leq t - 1$.

(4) For any $M \in M_{st}$ and $M \in (\mathcal{I}_s, \mathcal{I}_{s+1}]$, by Remark 5, we know that there is at least one $A \subseteq E, A \in \mathcal{B}(M)$ holds. In addition, by Property 2, $M_A$ with $\mathcal{B}(M_A) = \{A\}$ is an atom in $[\mathcal{I}_{s-1}, \mathcal{I}_s]$. In another word, $\mathcal{B}(M)$ contains at least $\mathcal{B}(M_A)$ as its real subset. Let $M_1, M_2$ be two maximal elements containing $A$ in $(\{M' | \mathcal{B}(M') \subseteq \mathcal{B}(M) \text{ and } M' \in [\mathcal{I}_{s-1}, \mathcal{I}_s]\}, \leq)$. Then $M_1||M_2$ and $M_1, M_2 \in M_{s(t-k)}$ where $1 \leq t - k = |\mathcal{B}(M_1)| = |\mathcal{B}(M_2)| \leq t - 1$.

(5) Unfortunately, by the method above, we could not pledged that two elements in $T_{st}^{(t-k)}$ are non-isomorphism though we could find all of them.

3. Method for Finding the Elements

In this section, we shall sketchily write the concrete process of the method for finding the whole members of $\mathcal{M}(E)$.

Based on Property 2, if the constituent elements of $[\mathcal{I}_{s-1}, \mathcal{I}_s]$ is obtained for $s = 1, \ldots, |E|$, then $\mathcal{M}(E)$ will be. Before the process to find the elements in $[\mathcal{I}_{s-1}, \mathcal{I}_s]$, we should notice that $|\mathcal{B}(M)| = l_s(M)$ holds for $\forall M \in [\mathcal{I}_{s-1}, \mathcal{I}_s]$ according to Corollary 2, where $l_s$ stands for the length of $M$ in $([\mathcal{I}_{s-1}, \mathcal{I}_s], \leq)$. In view of the property of $|\mathcal{B}(M)|$ that can not be changed at all for any $M' \in \mathcal{M}(E)$, it gets that for two different elements $M_1, M_2 \in [\mathcal{I}_{s-1}, \mathcal{I}_s]$, there is: $l_s(M_1) = l_s(M_2) \iff M_1||M_2$.

The following is a method to find the members in $[\mathcal{I}_{s-1}, \mathcal{I}_s]$.

Let us give a subscript $i$ for an element in $E, (i = 1, 2, \ldots, |E| = k)$, i.e. $E = \{a_1, a_2, \ldots, a_k\}$. Besides, the elements in $E$ are ordered by lexicographic
\(a_1 < a_2 < \cdots < a_k\). Furthermore, for any \(D_1 = \{a_{g_1}, a_{g_2}, \ldots, a_{g_s}\}\), \(D_2 = \{a_{f_1}, a_{f_2}, \ldots, a_{f_t}\} \subseteq E\) where \(g_1 < g_2 < \cdots < g_s\), \(f_1 < f_2 < \cdots < f_t\), we define \(D_1 < D_2 \Leftrightarrow "s < t"\) or \(s = t, g_1 = f_1, g_2 = f_2, \ldots, g_i = f_i, g_{i+1} < f_{i+1}"\). Then \(2^E\) which is denoted the whole subsets of \(E\) is ordered by the above binary relation \(<\).

We proceed by induction on length \(l_s\) in \([\mathcal{I}_s, \mathcal{I}_{s+1}]\) to get the elements in \([\mathcal{I}_{s-1}, \mathcal{I}_s]\) for some \(s \in \{0, 1, \ldots, |E| - 1\}\). Considering with Property 2, \(l_s = 0\) and \(l_s = 1\) are done and by Corollary 2, \(l_s = C_s^E\) is done because it has only one element \(\mathcal{I}_s\) satisfying \(l_s = C_s^E\). Besides, \(\mathcal{B}(\mathcal{M}_{s_1}) = \{B_g : B_g \in [\mathcal{I}_{s-1}, \mathcal{I}_s], l_s(B_g) = 1, g = 1, 2, \ldots, C_s^E\}\) is ordered trivially by the definition above. Therefore, suppose that the elements of \(l_s \leq t - 1\), i.e., the elements in \(\mathcal{M}_{sj} = \{M \in [\mathcal{I}_{s-1}, \mathcal{I}_s] | |\mathcal{B}(M)| = j\}\), are obtained and obviously, \(\mathcal{M}_{sj}\), is in order \((j = 1, 2, \ldots, t - 1)\).

We denote \(\mathcal{M}_st = \{M \in [\mathcal{I}_{s-1}, \mathcal{I}_s] | |\mathcal{B}(M)| = t\}\) and \(\mathcal{B}(\mathcal{M}_{st}) = \{\mathcal{B}(M)|M \in \mathcal{M}_{st}\}\) \((t = 1, 2, \ldots, C_s^E)\). Then \(\mathcal{M}_{s1}\) is the set of all atoms in \([\mathcal{I}_{s-1}, \mathcal{I}_s]\). Let \(\mathcal{B}(M_1) = \min(\mathcal{B}(\mathcal{M}_{s1})), \mathcal{B}(M_2) = \min(\mathcal{B}(\mathcal{M}_{s1}) \setminus \mathcal{B}(M_1))\). Consider \(|A_1 \cap A_2|\), where \(\mathcal{B}(M_1) = \{A_1\}, \mathcal{B}(M_2) = \{A_2\}\). By Corollary 3, if and only if \(|A_1 \cap A_2| = |A_1| - 1\), it has \(\mathcal{B}(M_1) \cup \mathcal{B}(M_2) \in \mathcal{B}(\mathcal{M}_{s2})\). Repeated application the above process, for \(\mathcal{B}(M_k) = \mathcal{B}(\mathcal{M}_{s1}) \setminus \{\mathcal{B}(M_2), \ldots, \mathcal{B}(M_{k-1})\}\) \((k = 2, \ldots, C_s^E)\). Finally, we can obtain \(\mathcal{B}(\mathcal{M}_{s2})\), and further, \(\mathcal{M}_{s2}\). Suppose we have obtain \(\mathcal{M}_{s(j-1)}\) \((j = 1, 2, \ldots, t - 1)\). Let \(\mathcal{B}(M_1) = \min(\mathcal{B}(\mathcal{M}_{s(t-1)})), \mathcal{B}(M_2) = \min(\mathcal{B}(\mathcal{M}_{s(t-1)}) \setminus \mathcal{B}(M_1))\). Considering \(|\mathcal{B}(M_1) \cap \mathcal{B}(M_2)|\), then by Property 4 and Remark 6, we can get all the matroids in \(\mathcal{M}_{st}\) produced by \(\mathcal{M}_{s(t-1)}, \mathcal{M}_{s(t-2)}, \ldots, \mathcal{M}_{s1}\), respectively. Besides, according to Remark 6, they are just constituted \(\mathcal{M}_{st}\).

In addition, \(|E| < \infty\), Property 2 and Corollary 2 taken together implies that the above process must be stopped in finite steps. Furthermore, \([\mathcal{I}_s, \mathcal{I}_{s+1}]\) is obtained. Consequently, \(\mathcal{M}(E)\) is obtained by finite steps.

The following example should serve to clarify the above method.

**Example.** Let \(E = \{a_1, a_2, a_3\}\). Then \(|E| = 3\).

Step 1. By Property 2, \(\mathcal{I}_0 = \{\emptyset\}, \mathcal{I}_1 = \{(a_1), (a_2), (a_3)\}, \mathcal{I}_2 = \{(a_1, a_2), (a_1, a_3), (a_2, a_3)\}\), and further, the atoms in \([\mathcal{I}_0, \mathcal{I}_1]\) are \{a_1\}, \{a_2\}
and \{a_3\}; the atoms in \([\mathcal{I}_1, \mathcal{I}_2]\) are \{(a_1, a_2)\}, \{(a_1, a_3)\}, \{(a_2, a_3)\}. Besides, \mathcal{I}_2 \prec \mathcal{I}_3 is obvious.

Step 2. By the method, in \([\mathcal{I}_0, \mathcal{I}_1]\), \mathcal{M}_{s_2} = \{(a_1), (a_2)\}, \{(a_1), (a_3)\}, \{(a_2), (a_3)\}\}. \mathcal{M}_{s_3} = \mathcal{I}_1, where \(s = 1\).

Step 3. By the method, in \([\mathcal{I}_1, \mathcal{I}_2]\), the elements such that the length is 2 are \{(a_1, a_2), (a_1, a_3)\}, \{(a_1, a_2), (a_2, a_3)\} and \{(a_1, a_3), (a_2, a_3)\}. That of the length 3 is \mathcal{I}_2.

Step 4. \mathcal{M}(E) = \{\mathcal{I}_0 = \emptyset, \mathcal{I}_3 = \{(a_1, a_2, a_3)\}, \mathcal{I}_1 = \{(a_1), (a_2), (a_3)\}, \mathcal{I}_2 = \{(a_1, a_2), (a_1, a_3), (a_2, a_3)\}, \{(a_1), (a_2)\}, \{(a_1), (a_3)\}, \{(a_1, a_2)\}, \{(a_2), (a_3)\}, \{(a_1, a_2), (a_1, a_3)\}, \{(a_1, a_2), (a_2, a_3)\}, \{(a_1, a_3), (a_2, a_3)\}\}.

References


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