ON WEAKLY PRECONTINUOUS FUNCTIONS IN BITOPOLOGICAL SPACES

BY

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Abstract. As a generalization of precontinuous functions, we introduce the notion of weakly precontinuous functions in bitopological spaces and obtain several characterizations and some properties of weakly precontinuous functions.

1. Introduction

The notion of preopen sets due to Mashhour et al. [16] plays a significant role in general topology. Janković [6] defined the notion of almost weakly continuous functions. Almost weakly continuous functions are further studied in [5], [21] and [17]. In [18] and [19], Paul and Bhattacharyya introduced and studied quasi precontinuous functions. It is shown in [17] that quasi-precontinuity is equivalent with almost weak continuity and weak precontinuity. In [7], [10] and [15], the concept of precontinuous functions in bitopological spaces is introduced and investigated.

In this paper we introduce the notion of weakly precontinuous functions in bitopological spaces and investigate the properties of these functions.

2. Preliminaries

Throughout the present paper, \((X, \tau_1, \tau_2)\) (resp. \((X, \tau)\)) denotes a bitopological (resp. topological) space. Let \((X, \tau)\) be a topological space and \(A\) a subset
of \( X \). The closure of \( A \) and the interior of \( A \) are denoted by \( \text{Cl}(A) \) and \( \text{Int}(A) \), respectively. Let \( (X, \tau_1, \tau_2) \) be a bitopological space and \( A \) a subset of \( X \). The closure of \( A \) and the interior of \( A \) with respect to \( \tau_i \) are denoted by \( i\text{Cl}(A) \) and \( i\text{Int}(A) \), respectively, for \( i = 1, 2 \).

**Definition 2.1.** A subset \( A \) of a bitopological space \( (X, \tau_1, \tau_2) \) is said to be

1. \((i, j)\)-regular open ([1]) if \( A = i\text{Int}(j\text{Cl}(A)) \), where \( i \neq j \), \( i, j = 1, 2 \),
2. \((i, j)\)-regular closed ([3]) if \( A = i\text{Cl}(j\text{Int}(A)) \), where \( i \neq j \), \( i, j = 1, 2 \),
3. \((i, j)\)-semi-open ([2]) if \( A \subset j\text{Cl}(i\text{Int}(A)) \), where \( i \neq j \), \( i, j = 1, 2 \),
4. \((i, j)\)-preopen ([7]) if \( A \subset i\text{Int}(j\text{Cl}(A)) \), where \( i \neq j \), \( i, j = 1, 2 \),
5. \((i, j)\)-\(\alpha\)-open ([8]) if \( A \subset i\text{Int}(j\text{Cl}(i\text{Int}(A))) \), where \( i \neq j \), \( i, j = 1, 2 \).

The complement of an \((i, j)\)-preopen set is said to be \((i, j)\)-preclosed ([10], [15]). A subset \( A \) is \((i, j)\)-preclosed if \( i\text{Cl}(j\text{Int}(A)) \subset A \).

**Lemma 2.1.** Let \( (X, \tau_1, \tau_2) \) be a bitopological space and \( \{A_\lambda : \lambda \in \Lambda \} \) a family of subsets of \( X \).

1. If \( A_\lambda \) is \((i, j)\)-preopen for each \( \lambda \in \Lambda \), then \( \bigcup_{\lambda \in \Lambda} A_\lambda \) is \((i, j)\)-preopen,
2. If \( A_\lambda \) is \((i, j)\)-preclosed for each \( \lambda \in \Lambda \), then \( \bigcap_{\lambda \in \Lambda} A_\lambda \) is \((i, j)\)-preclosed.

**Proof.** (1) The proof follows from Theorem 4.2 of [10] and Theorem 3.2 of [15].

(2) This is an immediate consequence of (1).

**Definition 2.2.** Let \( A \) be a subset of a bitopological space \( (X, \tau_1, \tau_2) \).

1. The \((i, j)\)-preclosure ([15]) of \( A \), denoted by \((i, j)\)-pCl\( (A) \), is defined by the intersection of all \((i, j)\)-preclosed sets containing \( A \).
2. The \((i, j)\)-preinterior of \( A \), denoted by \((i, j)\)-pInt\( (A) \), is defined by the union of all \((i, j)\)-preopen sets contained in \( A \).

**Lemma 2.2.** Let \( (X, \tau_1, \tau_2) \) be a bitopological space and \( A \) a subset of \( X \).

1. \((i, j)\)-pInt\( (A) \) is \((i, j)\)-preopen,
2. \((i, j)\)-pCl\( (A) \) is \((i, j)\)-preclosed,
(3) \( A \) is \((i, j)\)-preopen if and only if \( A = (i, j)\)-pInt\((A)\),
(4) \( A \) is \((i, j)\)-preclosed if and only if \( A = (i, j)\)-pCl\((A)\).

**Proof.** (1) and (2) follow from Lemma 2.1.
(3) and (4) follow from (1) and (2).

**Lemma 2.3.** For any subset \( A \) of a bitopological space \( (X, \tau_1, \tau_2) \), \( x \in (i, j)\)-pCl\((A)\) if and only if \( U \cap A \neq \emptyset \) for every \((i, j)\)-preopen set \( U \) containing \( x \).

**Lemma 2.4.** Let \((X, \tau_1, \tau_2)\) be a bitopological space and \( A \) a subset of \( X \).
(1) \( X - (i, j)\)-pInt\((A) = (i, j)\)-pCl\((X - A)\),
(2) \( X - (i, j)\)-pCl\((A) = (i, j)\)-pInt\((X - A)\).

**Proof.** (1) By Lemma 2.2, \((i, j)\)-pCl\((A)\) is \((i, j)\)-preclosed. Then \( X - (i, j)\)-pCl\((A)\) is \((i, j)\)-preopen. On the other hand, \( X - (i, j)\)-pCl\((X - A) \subset A \) and hence \( X - (i, j)\)-pCl\((X - A) \subset (i, j)\)-pInt\((A)\). Conversely, let \( x \in (i, j)\)-pInt\((A)\). Then there exists an \((i, j)\)-preopen set \( G \) such that \( x \in G \subset A \). Then \( X - G \) is \((i, j)\)-preclosed and \( X - A \subset X - G \). Since \( x \notin X - G \), \( x \notin (i, j)\)-pCl\((X - A)\) and hence \( (i, j)\)-pInt\((A) \subset X - (i, j)\)-pCl\((X - A)\). Therefore, \( X - (i, j)\)-pInt\((A) = (i, j)\)-pCl\((X - A)\).
(2) This follows from (1) immediately.

**Definition 2.3.** Let \((X, \tau_1, \tau_2)\) be a bitopological space and \( A \) a subset of \( X \). A point \( x \) of \( X \) is said to be in the \((i, j)\)-\(\theta\)-closure \((\mathbb{1})\) of \( A \), denoted by \((i, j)\)-Cl\(\theta\)(\(A\)), if \( A \cap j\text{Cl}(U) \neq \emptyset \) for every \( \tau_i\)-open set \( U \) containing \( x \), where \( i, j = 1, 2 \) and \( i \neq j \).

A subset \( A \) of \( X \) is said to be \((i, j)\)-\(\theta\)-closed if \( A = (i, j)\)-Cl\(\theta\)(\(A\)). A subset \( A \) of \( X \) is said to be \((i, j)\)-\(\theta\)-open if \( X - A \) is \((i, j)\)-\(\theta\)-closed. The \((i, j)\)-\(\theta\)-interior of \( A \), denoted by \((i, j)\)-Int\(\theta\)(\(A\)), is defined as the union of all \((i, j)\)-\(\theta\)-open sets contained in \( A \). Hence \( x \in (i, j)\)-Int\(\theta\)(\(A\)) if and only if there exists a \( \tau_i\)-open set \( U \) containing \( x \) such that \( x \in U \subset j\text{Cl}(U) \subset A \).

**Lemma 2.5.** For a subset \( A \) of a bitopological space \((X, \tau_1, \tau_2)\), the following properties hold:
Lemma 2.6. (Kariofillis [11]). Let $(X, \tau_1, \tau_2)$ be a bitopological space. If $U$ is a $\tau_j$-open set of $X$, then $(i, j)-\text{Cl}_\theta(U) = i\text{Cl}(U)$.

Definition 2.4. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be

1. $(i, j)$-semi-continuous ([2]) if $f^{-1}(V)$ is $(i, j)$-semi-open in $X$ for each $\sigma_i$-open set $V$ of $Y$,
2. $(i, j)$-precontinuous ([10], [15]) if $f^{-1}(V)$ is $(i, j)$-preopen in $X$ for each $\sigma_i$-open set $V$ of $Y$,
3. $(i, j)$-weakly continuous ([4]) if for each $x \in X$ and each $\sigma_i$-open set $V$ of $Y$ containing $f(x)$, there exists a $\tau_i$-open set $U$ containing $x$ such that $f(U) \subset j\text{Cl}(V)$.

Definition 2.5. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be $(i, j)$-weakly precontinuous if for each $x \in X$ and each $\sigma_i$-open set $V$ of $Y$ containing $f(x)$, there exists an $(i, j)$-preopen set $U$ containing $x$ such that $f(U) \subset j\text{Cl}(V)$.

A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be pairwise weakly continuous (resp. pairwise weakly precontinuous) if $f$ is weakly $(1,2)$-continuous and weakly $(2,1)$-continuous (resp. weakly $(1,2)$-precontinuous and weakly $(2,1)$-precontinuous).

Remark 2.1. Since every $\tau_i$-open set is $(i, j)$-preopen ([15]), every $(i, j)$-weakly continuous function is $(i, j)$-weakly precontinuous for $i, j = 1, 2$ and $i \neq j$. The converse is not true.

3. Characterizations

Theorem 3.1. For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

1. $f$ is $(i, j)$-weakly precontinuous;
2. $(i, j)-\text{pCl}(f^{-1}(j\text{Int}(i\text{Cl}(B)))) \subset f^{-1}(i\text{Cl}(B))$ for every subset $B$ of $Y$;
(3) \((i, j)\)-pCl\((f^{-1}(\text{Int}(F))) \subset f^{-1}(F)\) for every \((i, j)\)-regular closed set \(F\) of \(Y\);

(4) \((i, j)\)-pCl\((f^{-1}(V)) \subset f^{-1}(\text{Cl}(V))\) for every \(\sigma_j\)-open set \(V\) of \(Y\);

(5) \(f^{-1}(V) \subset (i, j)\)-pInt\((f^{-1}(\text{Cl}(V)))\) for every \(\sigma_1\)-open set \(V\) of \(Y\).

**Proof.** (1) \(\Rightarrow\) (2): Let \(B\) be any subset of \(Y\). Suppose that \(x \in X - f^{-1}(\text{Cl}(B))\). Then \(f(x) \in Y - i\text{Cl}(B)\) and there exists a \(\sigma_1\)-open set \(V\) of \(Y\) containing \(f(x)\) such that \(V \cap B = \emptyset\). Therefore, \(V \cap \text{Int}(i\text{Cl}(B)) = \emptyset\) and hence \(\text{Int}(i\text{Cl}(B)) \subset V\). Therefore, there exists an \((i, j)\)-preopen set \(U\) containing \(x\) such that \(f(U) \subset j\text{Cl}(V)\). Hence, we have \(U \cap f^{-1}(\text{Int}(i\text{Cl}(B))) = \emptyset\) and \(x \in X - (i, j)\)-pCl\((f^{-1}(\text{Int}(i\text{Cl}(B))))\) by Lemma 2.3. Thus, we obtain \((i, j)\)-pCl\((f^{-1}(\text{Int}(i\text{Cl}(B)))) \subset f^{-1}(\text{Cl}(B))\).

(2) \(\Rightarrow\) (3): Let \(F\) be an \((i, j)\)-regular closed set of \(Y\). Then \(F = i\text{Cl}(j\text{Int}(F))\) and we have \((i, j)\)-pCl\((f^{-1}(j\text{Int}(F))) = (i, j)\)-pCl\((f^{-1}(j\text{Int}(i\text{Cl}(j\text{Int}(F))))) \subset f^{-1}(i\text{Cl}(j\text{Int}(F))) = f^{-1}(F)\).

(3) \(\Rightarrow\) (4): Let \(V\) be a \(\sigma_j\)-open set of \(Y\). Then \(i\text{Cl}(V)\) is \((i, j)\)-regular closed. Then we obtain \((i, j)\)-pCl\((f^{-1}(V)) \subset (i, j)\)-pCl\((f^{-1}(\text{Int}(i\text{Cl}(V)))) \subset f^{-1}(i\text{Cl}(V))\).

(4) \(\Rightarrow\) (5): Let \(V\) be a \(\sigma_1\)-open set of \(Y\). Then \(Y - j\text{Cl}(V)\) is \(\sigma_j\)-open and we have \((i, j)\)-pCl\((Y - j\text{Cl}(V)) \subset f^{-1}(i\text{Cl}(Y - j\text{Cl}(V)))\) and hence \(X - (i, j)\)-pInt\((f^{-1}(j\text{Cl}(V))) \subset X - f^{-1}(\text{Int}(j\text{Cl}(V))) \subset X - f^{-1}(V)\). Therefore, we obtain \(f^{-1}(V) \subset (i, j)\)-pInt\((f^{-1}(j\text{Cl}(V))))\).

(5) \(\Rightarrow\) (1): Let \(x \in X\) and \(V\) be a \(\sigma_1\)-open set containing \(f(x)\). We have \(x \in f^{-1}(V) \subset (i, j)\)-pInt\((f^{-1}(j\text{Cl}(V)))\). Put \(U = (i, j)\)-pInt\((f^{-1}(j\text{Cl}(V))))\). By Lemma 2.2, \(U\) is \((i, j)\)-preopen set containing \(x\) and \(f(U) \subset j\text{Cl}(V)\). This shows that \(f\) is \((i, j)\)-weakly precontinuous.

**Remark 3.1.** Let \(\tau = \tau_1 = \tau_2\) and \(\sigma = \sigma_1 = \sigma_2\). Then by Theorem 3.1 we obtain the results for a function \(f : (X, \tau) \to (Y, \sigma)\) established in Theorems 3.1 and 3.3 of [21] and Theorem 1 of [18].

**Theorem 3.2.** For a function \(f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)\), the following properties are equivalent:

(1) \(f\) is \((i, j)\)-weakly precontinuous;

(2) \(((i, j)\)-pCl\((A) \subset (i, j)\)-Cl\(_\theta\)(\(f(A)\)) for every subset \(A\) of \(X\);

(3) \((i, j)\)-pCl\((f^{-1}(B)) \subset f^{-1}((i, j)\)-Cl\(_\theta\)(\(B)\)) for every subset \(B\) of \(Y\);
(4) \( (i,j)\)-\(p\text{Cl}(f^{-1}(j\text{Int}((i,j)\text{-}\text{Cl}_\theta(B)))) \subset f^{-1}((i,j)\text{-}\text{Cl}_\theta(B)) \) for every subset \( B \) of \( Y \).

**Proof.** (1) \( \Rightarrow \) (2): Assume that \( f \) is \((i,j)\)-weakly precontinuous. Let \( A \) be any subset of \( X \), \( x \in (i,j)\)-\(p\text{Cl}(A) \) and \( V \) be a \( \sigma_i \)-open set of \( Y \) containing \( f(x) \). Then, there exists an \((i,j)\)-preopen set \( U \) containing \( x \) such that \( f(U) \subset j\text{Cl}(V) \). Since \( x \in (i,j)\)-\(p\text{Cl}(A) \), by Lemma 2.3 we obtain \( U \cap A \neq \emptyset \) and hence \( \emptyset \neq f(U) \cap f(A) \subset j\text{Cl}(V) \cap f(A) \). Therefore, we obtain \( f(x) \in (i,j)\)-\(\text{Cl}_\theta(f(A)) \).

(2) \( \Rightarrow \) (3): Let \( B \) be any subset of \( Y \). Then we have \( f((i,j)\text{-}\text{Cl}(f^{-1}(B))) \subset (i,j)\text{-}\text{Cl}_\theta(f(f^{-1}(B))) \subset (i,j)\text{-}\text{Cl}_\theta(B) \) and hence \((i,j)\)-\(\text{Cl}(f^{-1}(B)) \subset f^{-1}((i,j)\text{-}\text{Cl}_\theta(B)) \).

(3) \( \Rightarrow \) (4): Let \( B \) be any subset of \( Y \). Since \((i,j)\)-\(\text{Cl}_\theta(B) \) is \( \sigma_i \)-closed in \( Y \), by Lemma 2.6 \((i,j)\)-\(p\text{Cl}(f^{-1}(j\text{Int}((i,j)\text{-}\text{Cl}_\theta(B)))) \subset f^{-1}((i,j)\text{-}\text{Cl}_\theta(j\text{Int}((i,j)\text{-}\text{Cl}_\theta(B)))) = f^{-1}(i\text{Cl}(j\text{Int}((i,j)\text{-}\text{Cl}_\theta(B)))) \subset f^{-1}(i\text{Cl}((i,j)\text{-}\text{Cl}_\theta(B)))) = f^{-1}((i,j)\text{-}\text{Cl}_\theta(B))).

(4) \( \Rightarrow \) (1): Let \( V \) be any \( \sigma_j \)-open set of \( Y \). Then by Lemma 2.6, \( V \subset j\text{Int}(i\text{Cl}(V)) = j\text{Int}((i,j)\text{-}\text{Cl}_\theta(V)) \) and we have \((i,j)\)-\(p\text{Cl}(f^{-1}(V)) \subset (i,j)\)-\(p\text{Cl}(f^{-1}(j\text{Int}((i,j)\text{-}\text{Cl}_\theta(V)))) \subset f^{-1}((i,j)\text{-}\text{Cl}_\theta(V)) \) and hence \((i,j)\)-\(p\text{Cl}(f^{-1}(V)) \subset f^{-1}((i,j)\text{-}\text{Cl}_\theta(V)) \). It follows from Theorem 3.1 that \( f \) is \((i,j)\)-weakly precontinuous.

**Remark 3.2.** By Theorem 3.2, we obtain the results established in Theorem 3.3 of [21].

**Definition 3.1.** A bitopological space \((X, \tau_1, \tau_2) \) is said to be \((i,j)\)-regular ([12]) if for each \( x \in X \) and each \( \tau_i \)-open set \( U \) containing \( x \), there exists a \( \tau_i \)-open set \( V \) such that \( x \in V \subset j\text{Cl}(V) \subset U \).

**Lemma 3.1.** (Popa and Noiri [22]) If a bitopological space \((X, \tau_1, \tau_2) \) is \((i,j)\)-regular, then \((i,j)\)-\(\text{Cl}_\theta(F) = F \) for every \( \tau_i \)-closed set \( F \).

**Theorem 3.3.** Let \((Y, \sigma_1, \sigma_2) \) be an \((i,j)\)-regular bitopological space. For a function \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \), the following properties are equivalent:

1. \( f \) is \((i,j)\)-precontinuous;
(2) \( f^{-1}((i,j)\text{-Cl}_\theta(B)) \) is \((i,j)\text{-preclosed in } X \) for every subset \( B \) of \( Y \);

(3) \( f \) is \((i,j)\text{-weakly precontinuous};

(4) \( f^{-1}(F) \) is \((i,j)\text{-preclosed in } X \) for every \((i,j)\theta\text{-closed set } F \) of \( Y \);

(5) \( f^{-1}(V) \) is \((i,j)\text{-preopen in } X \) for every \((i,j)\theta\text{-open set } V \) of \( Y \).

**Proof.** (1) \( \Rightarrow \) (2): Let \( B \) be any subset of \( Y \). Since \((i,j)\text{-Cl}_\theta(B)\) is \( \sigma_i \)-closed in \( Y \), it follows from Theorem 4.3 of [15] that \( f^{-1}((i,j)\text{-Cl}_\theta(B)) \) is \((i,j)\text{-preclosed in } X \).

(2) \( \Rightarrow \) (3): Let \( B \) be any subset of \( Y \). Then we have

\[
(i,j)\text{-pCl}(f^{-1}(B)) \subset (i,j)\text{-pCl}(f^{-1}((i,j)\text{-Cl}_\theta(B))) = f^{-1}((i,j)\text{-Cl}_\theta(B)).
\]

By Theorem 3.2, \( f \) is \((i,j)\text{-weakly precontinuous}.

(3) \( \Rightarrow \) (4): Let \( F \) be any \((i,j)\theta\text{-closed set of } Y \). Then by Theorem 3.2, \((i,j)\text{-pCl}(f^{-1}(F)) \subset f^{-1}((i,j)\text{-Cl}_\theta(F)) = f^{-1}(F) \). Therefore, by Lemma 2.2, \( f^{-1}(F) \) is \((i,j)\text{-preclosed in } X \).

(4) \( \Rightarrow \) (5): Let \( V \) be any \((i,j)\theta\text{-open set of } Y \). By (4), \( f^{-1}(Y - V) = X - f^{-1}(V) \) is \((i,j)\text{-preclosed in } X \) and hence \( f^{-1}(V) \) is \((i,j)\text{-preopen in } X \).

(5) \( \Rightarrow \) (1): Since \( Y \) is \((i,j)\text{-regular}, by Lemma 3.1 \((i,j)\text{-Cl}_\theta(B) = B \) for every \( \sigma_i \text{-closed set } B \) of \( Y \) and hence every \( \sigma_i \text{-open set is } (i,j)\theta\text{-open. Therefore, } f^{-1}(V) \) is \((i,j)\text{-preopen for every } \sigma_i \text{-open set } V \) of \( Y \). By Theorem 4.3 of [15], \( f \) is \((i,j)\text{-precontinuous}.

**Remark 3.3.** By Theorem 3.3, we obtain the results established in Theorem 2.5 of [5] and Corollary 3.4 of [21].

4. Weak Precontinuity and Precontinuity

**Definition 4.1.** A function \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is said to be \((i, j)\text{-weakly}^* \text{ quasicontinuous} \) (briefly \( w^*\text{-q.c.} \)) ([22]) if for every \( \sigma_i \text{-open set } V \) of \( Y \), \( f^{-1}(j\text{Cl}(V) - V) \) is biclosed in \( X \).

**Theorem 4.1.** If a function \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is \((i, j)\text{-weakly precontinuous and } (i, j)\text{-}w^*\text{-q.c.}, then } f \) is \((i, j)\text{-precontinuous}.

**Proof.** Let \( x \in X \) and \( V \) be any \( \sigma_i \text{-open set of } Y \) containing \( f(x) \). Since \( f \) is \((i, j)\text{-weakly precontinuous, there exists an } (i, j)\text{-preopen set } U \) of \( X \) containing
x such that $f(U) \subset j\text{Cl}(V)$. Hence $x \notin f^{-1}(j\text{Cl}(V) - V)$. Therefore, $x \in U - f^{-1}(j\text{Cl}(V) - V) = U \cap (X - f^{-1}(j\text{Cl}(V) - V))$. Since $U$ is $(i, j)$-preopen and $X - f^{-1}(j\text{Cl}(V) - V)$ is biopen, by Theorem 4.1 of [10], $G = U \cap (X - f^{-1}(j\text{Cl}(V) - V))$ is $(i, j)$-preopen. Then $x \in G$ and $f(G) \subset V$. For, if $y \in G$, then $f(y) \notin j\text{Cl}(V) - V$ and hence $f(y) \in V$. Therefore, $f$ is $(i, j)$-precontinuous.

**Definition 4.2.** A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to have $(i, j)$-
\text{pInteriorty condition} if $(i, j)$-pInt$(f^{-1}(j\text{Cl}(V))) \subset f^{-1}(V)$ for every \(\sigma_i\)-open set $V$ of $Y$.

**Theorem 4.2.** If a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is $(i, j)$-weakly
\text{precontinuous} and satisfies the $(i, j)$-p\text{Interiority condition}, then $f$ is $(i, j)$-precontinuous.

**Proof.** Let $V$ be any $\sigma_i$-open set of $Y$. Since $f$ is $(i, j)$-weakly precontinuous, by Theorem 3.1, $f^{-1}(V) \subset (i, j)$-pInt$(f^{-1}(j\text{Cl}(V)))$. By the $(i, j)$-p\text{Interiorty condition} of $f$, we have $(i, j)$-pInt$(f^{-1}(j\text{Cl}(V))) \subset f^{-1}(V)$ and hence $f^{-1}(V) = (i, j)$-pInt$(f^{-1}(j\text{Cl}(V)))$. By Lemma 2.2, $f^{-1}(V)$ is $(i, j)$-preopen in $X$ and thus $f$ is $(i, j)$-precontinuous.

**Definition 4.3.** Let $(X, \tau_1, \tau_2)$ be a bitopological space and $A$ be a subset
of $X$. The $(i, j)$-\text{prefrontier} of $A$ is defined as follows: $(i, j)$-pFr$(A) = (i, j)$-
\text{pCl}(A) \cap (i, j)$-pCl$(X - A) = (i, j)$-pCl$(A) - (i, j)$-pInt$(A)$.

**Theorem 4.3.** The set of all points $x$ of $X$ at which a function $f : (X, \tau_1, \tau_2)
\to (Y, \sigma_1, \sigma_2)$ is not $(i, j)$-weakly precontinuous is identical with the union of the
$(i, j)$-\text{prefrontiers} of the inverse images of the $\sigma_j$-closure of $\sigma_i$-open sets of $Y$
containing $f(x)$.

**Proof.** Let $x$ be a point of $X$ at which $f(x)$ is not $(i, j)$-weakly precontinuous.
Then, there exists a $\sigma_i$-open set $V$ of $Y$ containing $f(x)$ such that $U \cap (X - f^{-1}(j\text{Cl}(V))) \neq \emptyset$ for every $(i, j)$-preopen set $U$ of $X$ containing $x$. By Lemma
2.3, $x \in (i, j)$-pCl$(X - f^{-1}(j\text{Cl}(V)))$. Since $x \in f^{-1}(j\text{Cl}(V))$, we have $x \in (i, j)$-
pCl$(f^{-1}(j\text{Cl}(V)))$ and hence $x \in (i, j)$-pFr$(f^{-1}(j\text{Cl}(V)))$.

Conversely, if $f$ is $(i, j)$-weakly precontinuous at $x$, then for each $\sigma_i$-open set
$V$ of $Y$ containing $f(x)$, there exists an $(i, j)$-preopen set $U$ containing $x$ such
that \( f(U) \subseteq j\text{Cl}(V) \) and hence \( x \in U \subseteq f^{-1}(j\text{Cl}(V)) \). Therefore, we obtain that \( x \in (i,j)\text{-pInt}(f^{-1}(j\text{Cl}(V))) \). This contradicts that \( x \in (i,j)\text{-pFr}(f^{-1}(j\text{Cl}(V))) \).

5. Weak Precontinuity and Almost Precontinuity

**Definition 5.1.** A function \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is said to be \((i,j)\text{-almost precontinuous}\) if for each \( x \in X \) and each \( \sigma_i\text{-open set} V \) containing \( f(x) \), there exists an \((i,j)\text{-preopen set} U \) of \( X \) containing \( x \) such that \( f(U) \subseteq i\text{Int}(j\text{Cl}(V)) \).

**Lemma 5.1.** A function \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is \((i,j)\text{-almost precontinuous} if and only if \( f^{-1}(V) \) is \((i,j)\text{-preopen for each} \((i,j)\text{-regular open set} V \) of \( Y \).

**Definition 5.2.** A bitopological space \((X, \tau_1, \tau_2)\) is said to be \((i,j)\text{-almost regular} ([23]) \) if for each \( x \in X \) and each \((i,j)\text{-regular open set} U \) containing \( x \), there exists an \((i,j)\text{-regular open set} V \) of \( X \) such that \( x \in V \subseteq j\text{Cl}(V) \subseteq U \).

**Theorem 5.1.** Let a bitopological space \((Y, \sigma_1, \sigma_2)\) be \((i,j)\text{-almost regular}. Then a function \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is \((i,j)\text{-almost precontinuous if and only if it is} \((i,j)\text{-weakly precontinuous}.

**Proof.** Necessity. This is obvious.

Sufficiency. Suppose that \( f \) is \((i,j)\text{-weakly precontinuous}. Let \( V \) be any \((i,j)\text{-regular open set of} Y \) and \( x \in f^{-1}(V) \). Then we have \( f(x) \in V \). By the almost \((i,j)\text{-regularity of} Y \), there exists an \((i,j)\text{-regular open set} V_0 \) of \( Y \) such that \( f(x) \in V_0 \subseteq j\text{Cl}(V_0) \subseteq V \). Since \( f \) is \((i,j)\text{-weakly precontinuous}, there exists an \((i,j)\text{-preopen set} U \) of \( X \) containing \( x \) such that \( f(U) \subseteq j\text{Cl}(V_0) \subseteq V \). This implies that \( x \in U \subseteq f^{-1}(V) \). Therefore, we have \( f^{-1}(V) \subseteq (i,j)\text{-pInt}(f^{-1}(V)) \) and hence \( f^{-1}(V) = (i,j)\text{-pInt}(f^{-1}(V)) \). By Lemma 2.2, \( f^{-1}(V) \) is \((i,j)\text{-preopen} and by Lemma 5.1 \( f \) is \((i,j)\text{-almost precontinuous}.

**Definition 5.3.** A bitopological space \((X, \tau_1, \tau_2)\) is said to be \textit{pairwise Hausdorff} or \textit{pairwise} \( T_2 \) ([12]) if for each pair of distinct points \( x \) and \( y \) of \( X \), there exist a \( \tau_i\text{-open set} U \) containing \( x \) and a \( \tau_j\text{-open set} V \) containing \( y \) such that \( U \cap V = \emptyset \) for \( i \neq j, i, j = 1,2 \).
Definition 5.4. A bitopological space \((X, \tau_1, \tau_2)\) is said to be pairwise pre-\(T_2\) if for each pair of distinct points \(x\) and \(y\) of \(X\), there exist a \((i, j)\)-preopen set \(U\) containing \(x\) and a \((j, i)\)-preopen set \(V\) containing \(y\) such that \(U \cap V = \emptyset\) for \(i \neq j, i, j = 1, 2\).

Theorem 5.2. Let \((X, \tau_1, \tau_2)\) be a bitopological space. If for each pair of distinct points \(x\) and \(y\) in \(X\), there exists a function \(f\) of \((X, \tau_1, \tau_2)\) into a pairwise \(T_2\) bitopological space \((Y, \sigma_1, \sigma_2)\) such that
\begin{enumerate}
  \item \(f(x) \neq f(y)\),
  \item \(f\) is \((i, j)\)-weakly precontinuous at \(x\), and
  \item \(f\) is \((j, i)\)-almost precontinuous at \(y\),
\end{enumerate}
then \((X, \tau_1, \tau_2)\) is pairwise pre-\(T_2\).

Proof. Let \(x\) and \(y\) be a pair of distinct points of \(X\). Since \(Y\) is pairwise \(T_2\), there exists a \(\sigma_i\)-open set \(U\) containing \(f(x)\) and a \(\sigma_j\)-open set \(V\) containing \(f(y)\) such that \(U \cap V = \emptyset\). Since \(U\) and \(V\) are disjoint, we have \(j\text{Cl}(U) \cap j\text{Int}(i\text{Cl}(V)) = \emptyset\). Since \(f\) is \((i, j)\)-weakly precontinuous at \(x\), there exists an \((i, j)\)-preopen set \(U_x\) of \(X\) containing \(x\) such that \(f(U_x) \subset j\text{Cl}(U)\). Since \(f\) is \((j, i)\)-almost precontinuous at \(y\), there exists a \((j, i)\)-preopen set \(U_y\) of \(X\) containing \(y\) such that \(f(U_y) \subset j\text{Int}(i\text{Cl}(V))\). Hence we have \(U_x \cap U_y = \emptyset\). This shows that \((X, \tau_1, \tau_2)\) is pairwise pre-\(T_2\).

6. Some Properties

Definition 6.1. A bitopological space \((X, \tau_1, \tau_2)\) is said to be pairwise Urysohn ([3]) if for each distinct points \(x, y\) of \(X\) there exist a \(\tau_i\)-open set \(U\) and a \(\tau_j\)-open set \(V\) such that \(x \in U, y \in V\) and \(j\text{Cl}(U) \cap i\text{Cl}(V) = \emptyset\) for \(i \neq j, i, j = 1, 2\).

Theorem 6.1. If \((Y, \sigma_1, \sigma_2)\) is a pairwise Urysohn and \(f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) is a pairwise weakly precontinuous injection, then \((X, \tau_1, \tau_2)\) is pairwise pre-\(T_2\).

Proof. Let \(x\) and \(y\) be any distinct points of \(X\). Then \(f(x) \neq f(y)\). Since \(Y\) is pairwise Urysohn, there exist a \(\tau_i\)-open set \(U\) and a \(\tau_j\)-open set \(V\)
such that \( f(x) \in U, f(y) \in V \) and \( j\text{Cl}(U) \cap i\text{Cl}(V) = \emptyset \). Hence \( f^{-1}(j\text{Cl}(U)) \cap f^{-1}(i\text{Cl}(V)) = \emptyset \). Therefore, \( (i, j)\text{-pInt}(f^{-1}(j\text{Cl}(U))) \cap (j, i)\text{-pInt}(f^{-1}(i\text{Cl}(V))) = \emptyset \). Since \( f \) is pairwise weakly precontinuous, by Theorem 3.1 \( x \in f^{-1}(U) \subseteq (i, j)\text{-pInt}(f^{-1}(j\text{Cl}(U))) \) and \( y \in f^{-1}(V) \subseteq (j, i)\text{-pInt}(f^{-1}(i\text{Cl}(V))) \). This implies that \((X, \tau_1, \tau_2)\) is pairwise pre\(T_2\).

**Remark 6.1.** By Theorem 6.1 we obtain the result established in Theorem 5.1 of [19].

**Definition 6.2.** A bitopological space \((X, \tau_1, \tau_2)\) is said to be pairwise connected ([20]) (resp. pairwise preconnected) if it cannot be expressed as the union of two nonempty disjoint sets \( U \) and \( V \) such that \( U \) is \( \tau_1\)-open and \( V \) is \( \tau_2\)-open (resp. \( U \) is \((i, j)\)-preopen and \( V \) is \((j, i)\)-preopen).

**Theorem 6.2.** If a function \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is a pairwise weakly precontinuous surjection and \((X, \tau_1, \tau_2)\) is pairwise preconnected, then \((Y, \sigma_1, \sigma_2)\) is pairwise connected.

**Proof.** Suppose that \((Y, \sigma_1, \sigma_2)\) is not pairwise connected. Then, there exist a \( \tau_1\)-open set \( U \) and a \( \tau_2\)-open set \( V \) such that \( U \neq \emptyset, V \neq \emptyset, U \cap V = \emptyset \) and \( U \cup V = Y \). Since \( f \) is surjective, \( f^{-1}(U) \) and \( f^{-1}(V) \) are nonempty. Moreover \( f^{-1}(U) \cap f^{-1}(V) = \emptyset \) and \( f^{-1}(U) \cup f^{-1}(V) = X \). Since \( f \) is pairwise weakly precontinuous, by Theorem 3.1 we have \( f^{-1}(U) \subseteq (i, j)\text{-pInt}(f^{-1}(j\text{Cl}(U))) \) and \( f^{-1}(V) \subseteq (j, i)\text{-pInt}(f^{-1}(i\text{Cl}(V))) \). Since \( U \) and \( V \) are \( \sigma_1\)-closed and \( \sigma_1\)-closed, respectively, we have \( f^{-1}(U) \subseteq (i, j)\text{-pInt}(f^{-1}(U)) \) and \( f^{-1}(V) \subseteq (j, i)\text{-pInt}(f^{-1}(V)) \). Hence \( f^{-1}(U) = (i, j)\text{-pInt}(f^{-1}(U)) \) and \( f^{-1}(V) = (j, i)\text{-pInt}(f^{-1}(V)) \). By Lemma 2.2, \( f^{-1}(U) \) is \((i, j)\)-preopen and \( f^{-1}(V) \) is \((j, i)\)-preopen in \((X, \tau_1, \tau_2)\). This shows that \((X, \tau_1, \tau_2)\) is not pairwise preconnected.

**Remark 6.2.** By Theorem 6.2, we obtain the result established in Theorem 2.12 of [5] and Theorem 4.7 of [19].

**Definition 6.3.** A subset \( K \) of a bitopological space \((X, \tau_1, \tau_2)\) is said to be \((i, j)\)-quasi \(H\)-closed relative to \( X \) ([1]) if for each cover \( \{U_\alpha : \alpha \in \Delta\} \) of \( K \) by \( \tau_1\)-open sets of \( X \), there exists a finite subset \( \Delta_0 \) of \( \Delta \) such that \( K \subseteq \cup\{j\text{Cl}(U_\alpha) : \alpha \in \Delta_0\} \).
Definition 6.4. A subset $K$ of a bitopological space $(X, \tau_1, \tau_2)$ is said to be $(i, j)$-\textit{precompact relative to} $X$ if every cover of $K$ by $(i, j)$-preopen sets of $X$ has a finite subcover.

Theorem 6.3. If $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is $(i, j)$-Weakly precontinuous and $K$ is $(i, j)$-precompact relative to $X$, then $f(K)$ is $(i, j)$-quasi $H$-closed relative to $Y$.

Proof. Let $K$ be $(i, j)$-precompact relative to $X$ and $\{V_\alpha : \alpha \in \Delta\}$ any cover of $f(K)$ by $\sigma_1$-open sets of $(Y, \sigma_1, \sigma_2)$. Then $f(K) \subset \bigcup \{V_\alpha : \alpha \in \Delta\}$ and so $K \subset \bigcup \{f^{-1}(V_\alpha) : \alpha \in \Delta\}$. Since $f$ is $(i, j)$-weakly precontinuous, by Theorem 3.1 we have $f^{-1}(V_\alpha) \subset (i, j)\mbox{-}p\text{Int}(f^{-1}(j\text{Cl}(V_\alpha)))$ for each $\alpha \in \Delta$. Therefore, $K \subset \bigcup \{(i, j)\mbox{-}p\text{Int}(f^{-1}(j\text{Cl}(V_\alpha))) : \alpha \in \Delta\}$. Since $K$ is $(i, j)$-precompact relative to $X$ and $(i, j)\mbox{-}p\text{Int}(f^{-1}(j\text{Cl}(V_\alpha)))$ is $(i, j)$-preopen for each $\alpha \in \Delta$, there exists a finite subset $\Delta_0$ of $\Delta$ such that $K \subset \bigcup \{(i, j)\mbox{-}p\text{Int}(f^{-1}(j\text{Cl}(V_\alpha))) : \alpha \in \Delta_0\}$. This implies that

$$f(K) \subset \bigcup \{f((i, j)\mbox{-}p\text{Int}(f^{-1}(j\text{Cl}(V_\alpha)))) : \alpha \in \Delta_0\}$$

$$\subset \bigcup \{f(f^{-1}(j\text{Cl}(V_\alpha))) : \alpha \in \Delta_0\} \subset \bigcup \{j\text{Cl}(V_\alpha) : \alpha \in \Delta_0\}.$$

Hence $f(K)$ is $(i, j)$-quasi $H$-closed relative to $Y$.

Remark 6.3. By Theorem 6.3, we obtain the result established in Theorem 2.1 of [5] and Theorem 4.8 of [19] which is a property of a function $f : (X, \tau) \rightarrow (Y, \sigma)$.

Theorem 6.4. If a function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is $(i, j)$-Weakly precontinuous and $(j, i)$-semi-continuous, then $f$ is $(i, j)$-weakly continuous.

Proof. Let $V$ be a $\sigma_1$-open set of $(Y, \sigma_1, \sigma_2)$. Since $f$ is $(i, j)$-weakly precontinuous, by Theorem 3.1 and Lemma 2.2 we obtain $f^{-1}(V) \subset (i, j)\mbox{-}p\text{Int}(f^{-1}(j\text{Cl}(V))) \subset \text{iInt}(j\text{Cl}((i, j)\mbox{-}p\text{Int}(f^{-1}(j\text{Cl}(V)))) \subset \text{iInt}(j\text{Cl}(f^{-1}(j\text{Cl}(V))))).$

Since $j\text{Cl}(V)$ is $\sigma_j$-closed and $f$ is $(j, i)$-semi-continuous, $f^{-1}(j\text{Cl}(V))$ is $(j, i)$-semi-closed and $\text{iInt}(j\text{Cl}(f^{-1}(j\text{Cl}(V)))) \subset f^{-1}(j\text{Cl}(V))$. Hence $f^{-1}(V) \subset \text{iInt}(j\text{Cl}(f^{-1}(j\text{Cl}(V)))) \subset \text{iInt}(f^{-1}(j\text{Cl}(V))).$ By Lemma 3.1 of [4], $f$ is $(i, j)$-weakly continuous.
Remark 6.4. By Theorem 6.4, we obtain the result established in Theorem 3.4 of [17] and Theorem 6.3 of [19] which is a property of a function \( f : (X, \tau) \to (Y, \sigma) \).

Lemma 6.1. (Khedr [13]) If \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is \((i, j)\)-weakly continuous and \((Y, \sigma_1, \sigma_2)\) is \((i, j)\)-regular, then \( f \) is \( i \)-continuous.

Corollary 6.1. If \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is \((i, j)\)-weakly precontinuous, \((j, i)\)-semi-continuous and \((Y, \sigma_1, \sigma_2)\) is \((i, j)\)-regular, then \( f \) is \( i \)-continuous.

Proof. This follows from Theorem 6.4 and Lemma 6.1.

Remark 6.5. By Corollary 6.1, we obtain the result established in Theorem 6.9 of [19] which is a property of a function \( f : (X, \tau) \to (Y, \sigma) \).

References


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