SOME COEFFICIENT INEQUALITIES FOR CERTAIN
SUBCLASSES OF ANALYTIC FUNCTIONS
WITH RESPECT TO $k$-SYMMETRIC POINTS

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Abstract. In the present paper, the authors introduce two new subclasses $M^{(k)}(\alpha)$ and $N^{(k)}(\alpha)$ of analytic functions with respect to $k$-symmetric points. Some coefficient inequalities for functions belonging to these classes and their subclasses with positive coefficients are provided.

1. Introduction

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

Let $M(\alpha)$ be the subclass of $A$ consisting of functions $f(z)$ which satisfy the inequality

$$\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} < \alpha, \quad (z \in U),$$

for some $\alpha (\alpha > 1)$. And let $N(\alpha)$ be the subclass of $A$ consisting of functions $f(z)$ which satisfy the inequality

$$\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} < \alpha, \quad (z \in U),$$
for some $\alpha$ ($\alpha > 1$). The classes $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$ were introduced and investigated recently by Owa and Nishiwaki [1] (see also Srivastava and Attiya [2]).

Motivated by $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$, we introduce the following two subclasses of analytic functions with respect to $k$-symmetric points, and obtain some interesting results.

A function $f(z) \in A$ is in the class $\mathcal{M}^{(k)}(\alpha)$ if
\[
\text{Re} \left\{ \frac{zf'(z)}{f_k(z)} \right\} < \alpha, \quad (z \in \mathcal{U}),
\]
where $\alpha > 1$, $k \geq 1$ is a fixed positive integer and $f_k(z)$ is defined by the following equality
\[
f_k(z) = \frac{1}{k} \sum_{\nu=0}^{k-1} \varepsilon^{-\nu} f(\varepsilon^\nu z), \quad (\varepsilon^k = 1; \ z \in \mathcal{U}). \quad (1.1)
\]
And a function $f(z) \in A$ is in the class $\mathcal{N}^{(k)}(\alpha)$ if and only if $zf'(z) \in \mathcal{M}^{(k)}(\alpha)$.

In the present paper, we shall provide some coefficient inequalities for functions belonging to the classes $\mathcal{M}^{(k)}(\alpha)$ and $\mathcal{N}^{(k)}(\alpha)$ and their subclasses with positive coefficients.

2. Main Results

**Theorem 1.** Let $\alpha > 1$. If $f(z) \in A$ satisfies
\[
\sum_{n=1}^{\infty} \left[ (nk + 1) + |nk + 1 - 2\alpha| \right] |a_{nk+1}| + \sum_{n=2; n \neq lk+1}^{\infty} 2n |a_n| \leq 2(\alpha - 1), \quad (2.1)
\]
then $f(z) \in \mathcal{M}^{(k)}(\alpha)$.

**Proof.** Suppose that $f(z) \in A$ with $\alpha > 1$, it suffices to show that
\[
\left| \frac{zf'(z)}{f_k(z)} \right| < \left| \frac{zf'(z)}{f_k(z)} - 2\alpha \right|, \quad (z \in \mathcal{U}).
\]

Let $M$ be denoted by
\[
M := \left| zf'(z) \right| - \left| zf'(z) - 2\alpha f_k(z) \right|
\]
\[
= \left| z + \sum_{n=2}^{\infty} na_n z^n \right| - \left| z + \sum_{n=2}^{\infty} na_n z^n - 2\alpha z - 2\alpha \sum_{n=2}^{\infty} a_n b_n z^n \right|,
\]
where
\[ b_n = \frac{1}{k} \sum_{\nu=0}^{k-1} \varepsilon^{(n-1)\nu}, \quad (\varepsilon^k = 1). \]

Thus, for \( |z| = r < 1 \), we have
\[
M \leq r + \sum_{n=2}^{\infty} n |a_n| r^n - \left[ (2\alpha - 1)r - \sum_{n=2}^{\infty} |n - 2\alpha b_n| |a_n| r^n \right] < \left\{ \sum_{n=2}^{\infty} |n + |n - 2\alpha b_n|| |a_n| - 2(\alpha - 1) \right\} r. \tag{2.2}
\]

From the definition of \( b_n \), we know
\[
b_n = \begin{cases} 
1, & n = lk + 1, \\
0, & n \neq lk + 1. 
\end{cases} \tag{2.3}
\]

Substituting (2.3) into inequality (2.2), we get
\[
M < \left\{ \sum_{n=1}^{\infty} [(nk + 1) + |nk + 1 - 2\alpha|] |a_{nk+1}| + \sum_{n=2}^{\infty} 2n |a_n| - 2(\alpha - 1) \right\} r. 
\]

From (2.1), we know that \( M < 0 \). Thus we have
\[
\text{Re} \left\{ \frac{zf'(z)}{f_k(z)} \right\} < \alpha, \quad (z \in \mathcal{U}),
\]
that is \( f(z) \in \mathcal{M}^{(k)}(\alpha) \). This completes the proof of Theorem 1.

Similarly, for the class \( \mathcal{N}^{(k)}(\alpha) \), we have

**Corollary 1.** Let \( \alpha > 1 \). If \( f(z) \in \mathcal{A} \) satisfies
\[
\sum_{n=1}^{\infty} (nk + 1)(nk + 1) + |nk + 1 - 2\alpha| |a_{nk+1}| + \sum_{n=2}^{\infty} 2n^2 |a_n| \leq 2(\alpha - 1),
\]
then \( f(z) \in \mathcal{N}^{(k)}(\alpha) \).

We now provide the necessary and sufficient coefficient conditions for the following two classes \( \mathcal{M}^{(k)}_1(\alpha) \) and \( \mathcal{N}^{(k)}_1(\alpha) \), which are subclasses with positive
coefficients of the classes $\mathcal{M}^{(k)}(\alpha)$ and $\mathcal{N}^{(k)}(\alpha)$, respectively.

$$\mathcal{M}_1^{(k)}(\alpha) = \left\{ f(z) \in \mathcal{M}^{(k)}(\alpha) : f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \text{ with } a_n \geq 0 \ (n \geq 2) \right\},$$

and

$$\mathcal{N}_1^{(k)}(\alpha) = \left\{ f(z) \in \mathcal{N}^{(k)}(\alpha) : f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \text{ with } a_n \geq 0 \ (n \geq 2) \right\}.$$

**Theorem 2.** Let $k \geq 2$, $1 < \alpha \leq k + 1$ and $f(z) \in A$, then $f(z) \in \mathcal{M}_1^{(k)}(\alpha)$ if and only if

$$\sum_{n=2}^{\infty} na_n - \alpha \sum_{l=1}^{\infty} a_{lk+1} \leq \alpha - 1.$$

**Proof.** In view of Theorem 1, we need only to prove the necessity. Let

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{M}_1^{(k)}(\alpha),$$

then $a_n \geq 0$ for $n \geq 2$ and

$$\text{Re} \left\{ \frac{zf'(z)}{f_k(z)} \right\} < \alpha,$$

this is equivalent to

$$\left| \frac{zf'(z)}{f_k(z)} \right| < \left| \frac{zf'(z)}{f_k(z)} - 2\alpha \right|,$$

or equivalently,

$$|zf'(z)| < |zf'(z) - 2\alpha f_k(z)|.$$

Hence

$$\left| 1 + \sum_{n=2}^{\infty} na_n z^{n-1} \right| < \left| 1 + \sum_{n=2}^{\infty} na_n z^{n-1} - 2\alpha - 2\alpha \sum_{l=1}^{\infty} a_{lk+1} z^l \right|.$$ 

Setting $z \to 1^-$, noting that $a_n \geq 0$ for $n \geq 2$ and $\alpha > 1$, we have

$$1 + \sum_{n=2}^{\infty} na_n \leq 2\alpha - 1 + 2\alpha \sum_{l=1}^{\infty} a_{lk+1} - \sum_{n=2}^{\infty} na_n,$$

that is,

$$\sum_{n=2}^{\infty} na_n - \alpha \sum_{l=1}^{\infty} a_{lk+1} \leq \alpha - 1.$$
Hence the proof of Theorem 2 is complete.

Similarly, for the class $N^{(k)}_1(\alpha)$, we have

**Corollary 2.** Let $k \geq 2$, $1 < \alpha \leq k + 1$ and $f(z) \in \mathcal{A}$, then $f(z) \in N^{(k)}_1(\alpha)$ if and only if

$$\sum_{n=2}^{\infty} n^2 a_n - \alpha \sum_{l=1}^{\infty} (lk + 1)a_{lk+1} \leq \alpha - 1.$$ 

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**References**
