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BEST EXTENSION OF THE HILBERT'S TYPE INEQUALITY WITH MULTI-PARAMETERS

BY

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Abstract. By introducing parameter $\lambda > 0$ and improving the weight function, we obtain a generalization of Hilbert's type inequality with the best constant factor. As its applications, we build its equivalent form and some particular results.

1. Introduction

If $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, f, g are non-negative functions such that $0<\int_0^\infty f^p(t)dt<\infty$ and $0 < \int_0^\infty g^q(t)dt < \infty$, then we have

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\frac{\pi}{p})}\Big\{\int_0^\infty f^p(t)dt\Big\}^{\frac{1}{p}}\Big\{\int_0^\infty g^q(t)dt\Big\}^{\frac{1}{q}}, \tag{1.1}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Inequality (1.1) is well known as Hardy-Hilbert's integral inequality [1], which is important in analysis and applications ([7]).

In 2002, Yang [8] gave a generalization of (1.1) by introducing a parameter $\lambda > 0$ as:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda+y^\lambda}dxdy < \frac{\pi}{\lambda\sin(\frac{\pi}{p})}\Big\{\int_0^\infty t^{(p-1)(1-\lambda)}f^p(t)dt\Big\}^{\frac{1}{p}}\Big\{\int_0^\infty t^{(q-1)(1-\lambda)}g^q(t)dt\Big\}^{\frac{1}{q}}, \tag{1.2}$$

where the constant factor $\frac{\pi}{\lambda\sin(\pi/p)}$ is the best possible.

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In 2004, by introducing some parameters and estimating the weight function, Yang [9] gave an extension of (1.1) with the best constant factor as:

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x^\lambda+y^\lambda}dxdy < \frac{\pi}{\lambda\sin(\frac{\pi}{r})}\Big\{\int_0^\infty x^{p(1-\frac{\lambda}{r})-1}f^p(x)dx\Big\}^{\frac{1}{p}}\Big\{\int_0^\infty x^{q(1-\frac{\lambda}{s})-1}g^q(x)dx\Big\}^{\frac{1}{q}}, \tag{1.3}$$

where the constant factor $\frac{\pi}{\lambda\sin(\pi/r)}$ is the best possible. Recently, [2, 3, 4, 5] considered some multiple extensions of (1.1).

In 2005, by introducing the β and a parameter λ Yang [11] provided a double series inequality with a parameter $\lambda > 0$ and function ϕ related to (λ, p, q) as:

If $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, λ, $\alpha > 0$, $0 < \phi_r \le 1$ $(r = p, q)$, $\phi_p + \phi_q = \lambda\alpha$ and $a_nb_n \ge 0$ satisfy $0 < \sum_{n=1}^\infty n^{p(1-\phi_p)-1} < \infty$ and $0 < \sum_{n=1}^\infty n^{q(1-\phi_q)-1} < \infty$ then one has

$$\sum_{n=1}^\infty\sum_{m=1}^\infty \frac{a_mb_n}{(m^\alpha+n^\alpha)^\lambda} < \frac{1}{\alpha}B(\frac{\phi_p}{\alpha},\frac{\phi_q}{\alpha})\Big\{\sum_{n=1}^\infty n^{p(1-\phi_q)-1}a_n^p\Big\}^{\frac{1}{p}}\Big\{n^{q(1-\phi_p)-1}b_n^q\Big\}^{\frac{1}{q}}, \tag{1.4}$$

where the constant factor $\frac{1}{\alpha}B(\frac{\phi_p}{\alpha},\frac{\phi_q}{\alpha})$ is the best possible.

Under the same condition with (1.1), we still have (see [1, Th.341]):

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}}dxdy < pq\Big\{\int_0^\infty f^p(t)dt\Big\}^{\frac{1}{p}}\Big\{\int_0^\infty g^q(t)dt\Big\}^{\frac{1}{q}}, \tag{1.5}$$

where the constant factor pq is the best possible. In particular, for $p = q = 2$, we have

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}}dxdy < 4\Big\{\int_0^\infty f^2(t)dt\Big\}^{\frac{1}{p}}\Big\{\int_0^\infty g^2(t)dt\Big\}^{\frac{1}{q}}. \tag{1.6}$$

In 2004, by introducing a parameter $\lambda > 2-\min\{p,q\}$ and the weight function, Yang [10, 12] gave two generations of (1.4) as:

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{\max\{x^\lambda,y^\lambda\}}dxdy < k_\lambda(p)\Big\{\int_0^\infty t^{(p-1)(2-\lambda)-1}f^p(t)dt\Big\}^{\frac{1}{p}} \times\Big\{\int_0^\infty t^{(q-1)(2-\lambda)-1}g^q(t)dt\Big\}^{\frac{1}{q}}, \tag{1.7}$$

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{\max\{x^\lambda,y^\lambda\}}dxdy < k_\lambda(p)\Big\{\int_0^\infty t^{1-\lambda}f^p(t)dt\Big\}^{\frac{1}{p}}\cdot\Big\{\int_0^\infty t^{1-\lambda}g^q(t)dt\Big\}^{\frac{1}{q}}, \tag{1.8}$$

where the constant factor $k_\lambda(p) = \frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)}$ is the best possible.

In this paper, by using the β function and obtaining the expression of the weight function, we give a new extension of (1.7) and (1.8) with parameters λ and functions–(ϕ_p, ψ_q) as (1.4). As applications, we also consider the equivalent form and some other particular results.

2. Some Lemmas

Lemma 2.1. *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\phi_p, \psi_q > 0$, $\lambda > 0$, $\phi_p + \psi_q = \lambda$, define the weight function $\omega_\lambda(p, x)$ as*

$$\omega_\lambda(p, x) = \int_0^\infty \frac{1}{\max\{x^\lambda, y^\lambda\}} \cdot \frac{x^{(p-1)(1-\phi_p)}}{y^{1-\psi_q}} dy, \quad x \in (0, \infty). \tag{2.1}$$

Then we have

$$\omega_\lambda(p, x) = \frac{\lambda}{\phi_p \psi_q} x^{p(1-\phi_p)-1}. \tag{2.2}$$

Proof. By (2.1), we have

$$\begin{aligned}
\omega_\lambda(p, x) &= \int_0^\infty \frac{1}{\max\{x^\lambda, y^\lambda\}} \cdot \frac{x^{(p-1)(1-\phi_p)}}{y^{1-\psi_q}} dy \\
&= \int_0^x \frac{1}{\max\{x^\lambda, y^\lambda\}} \cdot \frac{x^{(p-1)(1-\phi_p)}}{y^{1-\psi_q}} dy + \int_x^\infty \frac{1}{\max\{x^\lambda, y^\lambda\}} \cdot \frac{x^{(p-1)(1-\phi_p)}}{y^{1-\psi_q}} dy \\
&= \int_0^x \frac{1}{x^\lambda} \cdot \frac{x^{(p-1)(1-\phi_p)}}{y^{1-\psi_q}} dy + \int_x^\infty \frac{1}{y^\lambda} \cdot \frac{x^{(p-1)(1-\phi_p)}}{y^{1-\psi_q}} dy \\
&= \frac{\lambda}{\phi_p \psi_q} x^{p(1-\phi_p)-1}.
\end{aligned} \tag{2.3}$$

Hence, (2.2) is valid and the lemma is proved.

Note. By (2.3), we still have

$$\omega_\lambda(q, y) = \frac{\lambda}{\phi_p \psi_q} y^{q(1-\psi_q)-1}. \tag{2.4}$$

3. Main Results and Applications

Theorem 3.1. *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\phi_p, \psi_q > 0$, $\lambda > 0$, $\phi_p + \psi_q = \lambda$, $f, g \geq 0$ such that $0 < \int_0^\infty x^{p(1-\phi_p)-1} f^p(x) < \infty$, and $0 < \int_0^\infty x^{q(1-\psi_q)-1} g^q(x) dx < \infty$,* *then we have*

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{\max\{x^\lambda,y^\lambda\}}dxdy < \frac{\lambda}{\phi_p\psi_q}\Big\{\int_0^\infty x^{p(1-\phi_p)-1}f^p(x)dx\Big\}^{\frac{1}{p}}\Big\{\int_0^\infty x^{q(1-\psi_q)-1}g^q(x)dx\Big\}^{\frac{1}{q}}, \tag{3.1}$$

where the constant factor $\frac{rs}{\lambda}$ *is the best possible. In particular,*
(a) *for* $\phi_p=\psi_q=\frac{\lambda}{2}$, *we have*

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{\max\{x^\lambda,y^\lambda\}}dxdy < \frac{4}{\lambda}\Big\{\int_0^\infty x^{p(1-\frac{\lambda}{2})-1}f^p(x)dx\Big\}^{\frac{1}{p}}\Big\{\int_0^\infty x^{q(1-\frac{\lambda}{2})-1}g^q(x)dx\Big\}^{\frac{1}{q}}; \tag{3.2}$$

(b) *for* $\lambda=1$, *we have*

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}}dxdy < \frac{1}{\phi_p\psi_q}\Big\{\int_0^\infty x^{p\psi_q-1}f^p(x)dx\Big\}^{\frac{1}{p}}\Big\{\int_0^\infty x^{q\phi_p-1}g^q(x)dx\Big\}. \tag{3.3}$$

Proof. By Hölder's inequality and Lemma 1, we have

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{\max\{x^\lambda,y^\lambda\}}dxdy\\
&=\int_0^\infty\int_0^\infty\Big[\frac{f(x)}{(\max\{x^\lambda,y^\lambda\})^{\frac{1}{p}}}\cdot\frac{x^{(1-\phi_p)/q}}{y^{(1-\psi_q)/p}}\Big]\Big[\frac{g(y)}{(\max\{x^\lambda,y^\lambda\})^{\frac{1}{q}}}\cdot\frac{y^{(1-\psi_q)/p}}{x^{(1-\phi_p)/q}}\Big]dxdy\\
&\le\Big\{\int_0^\infty\Big[\int_0^\infty\frac{1}{\max\{x^\lambda,y^\lambda\}}\cdot\frac{x^{(p-1)(1-\phi_p)}}{y^{(1-\psi_q)}}dy\Big]f^p(x)dx\Big\}^{\frac{1}{p}}\\
&\quad\times\Big\{\int_0^\infty\Big[\int_0^\infty\frac{1}{\max\{x^\lambda,y^\lambda\}}\cdot\frac{y^{(q-1)(1-\psi_q)}}{x^{(1-\phi_p)}}dx\Big]g^q(y)dy\Big\}^{\frac{1}{q}}\\
&=\Big\{\int_0^\infty\omega_\lambda(p,x)f^p(x)dx\Big\}^{\frac{1}{p}}\Big\{\int_0^\infty\omega_\lambda(q,y)g^q(y)dy\Big\}^{\frac{1}{q}}.
\end{aligned}\tag{3.4}$$

If (3.4) takes the form of equality, then there exists constants A and B, such that they are not all zero and (see [6])

$$A\frac{f^p(x)}{\max\{x^\lambda,y^\lambda\}}\cdot\frac{x^{(p-1)(1-\phi_p)}}{y^{1-\psi_q}}=B\frac{g^q(y)}{\max\{x^\lambda,y^\lambda\}}\cdot\frac{y^{(q-1)(1-\psi_q)}}{x^{1-\phi_p}},\ \text{a.e. in } (0,\infty)\times(0,\infty).$$

We find that $Ax \cdot x^{p(1-\phi_p)-1} f^p(x) = By \cdot y^{q(1-\psi_q)-1} g^q(y)$, a.e. in $(0,\infty)\times(0,\infty)$. Hence there exists a constant C, such that

$$Ax \cdot x^{p(1-\phi_p)-1} f^p(x) = C = By \cdot y^{q(1-\psi_q)-1} g^q(y), \text{ a.e. in } (0,\infty).$$

Without losing the generality, suppose $A \neq 0$, we may get $x^{p(1-\phi_p)-1} f^p(x) = C/(Ax)$, a.e. in $(0,\infty)$, which contradicts $0 < \int_0^\infty x^{p(1-\phi_p)-1} f^p(x)dx < \infty$. Hence (3.4) takes a strict inequality as follows:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x^\lambda, y^\lambda\}} dxdy < \Big\{ \int_0^\infty \omega_\lambda(p,x) f^p(x)dx \Big\}^{\frac{1}{p}} \Big\{ \int_0^\infty \omega_\lambda(q,y) g^q(y)dy \Big\}^{\frac{1}{q}}. \tag{3.5}$$

In view of (2.2) and (2.4), we have (3.1).

If the constant factor $\frac{\lambda}{\phi_p \psi_q}$ in (3.1) is not the best possible, then there exists a positive constant K (with $K < \frac{\lambda}{\phi_p \psi_q}$) and exists $a > 0$. We have

$$\int_a^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x^\lambda, y^\lambda\}} dxdy < K\Big\{ \int_a^\infty x^{p(1-\phi_p)-1} f^p(x)dx \Big\}^{\frac{1}{p}} \Big\{ \int_a^\infty x^{q(1-\psi_q)-1} g^q(x)dx \Big\}^{\frac{1}{q}}. \tag{3.6}$$

For $\varepsilon > 0$ small enough ($\varepsilon < p\phi_p$) and $0 < b < a$, setting f_ε and g_ε as:

$$f_\varepsilon(x) = g_\varepsilon(x) = 0, \quad x \in (0,b); \quad f_\varepsilon = x^{-1-\frac{\varepsilon}{p}+\phi_p}, \quad g_\varepsilon = x^{-1-\frac{\varepsilon}{q}+\psi_q}, \quad x \in [b,\infty),$$

then we find

$$\int_a^\infty \int_b^\infty \frac{f_\varepsilon(x)\cdot g_\varepsilon(y)}{\max\{x^\lambda, y^\lambda\}} dxdy = \int_a^\infty \int_b^\infty \frac{x^{-1-\frac{\varepsilon}{p}+\phi_p} \cdot y^{-1-\frac{\varepsilon}{q}+\psi_q}}{\max\{x^\lambda, y^\lambda\}} dxdy. \tag{3.7}$$

In (3.7), for $b \to 0^+$, by (3.6), we have

$$\frac{1}{a^\varepsilon}\Big[(\psi_q - \frac{\varepsilon}{q})^{-1} + (\phi_p + \frac{\varepsilon}{q})^{-1}\Big] = \varepsilon \int_a^\infty \int_0^\infty \frac{f_\varepsilon(x) g_\varepsilon(y)}{\max\{x^\lambda, y^\lambda\}} dxdy \leq \frac{K}{a^\varepsilon}.$$

For $\varepsilon^+ \to 0$, by [1] (see [1, Th.342 remark]), it follows that $\frac{\lambda}{\phi_p\psi_q} \leq K$, which contradicts the fact that $K < \frac{\lambda}{\phi_p\psi_q}$. Hence the constant factor $\frac{\lambda}{\phi_p\psi_q}$ in (3.1) is the best possible. The theorem is proved.

Theorem 3.2. *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\phi_p, \psi_q > 0$, $\lambda > 0$, $\phi_p + \psi_q = \lambda$, $f \geq 0$ *such that* $0 < \int_0^\infty x^{p(1-\phi_p)-1} f^p(x)dx < \infty$, *then we have*

$$\int_0^\infty y^{p\psi_q - 1}\Big[\int_0^\infty \frac{f(x)}{\max\{x^\lambda, y^\lambda\}}dx\Big]^p dy < \Big(\frac{\lambda}{\phi_p\psi_q}\Big)^p \int_0^\infty x^{p(1-\phi_p)-1} f^p(x)dx, \quad (3.8)$$

where the constant $(\frac{\lambda}{\phi_p\psi_q})^p$ *is the best possible. Inequality* (3.8) *is equivalent to* (3.1)*. In particular,*
(a) *for* $\phi_p = \psi_q = \frac{\lambda}{2}$, *we have*

$$\int_0^\infty y^{\frac{p\lambda}{2} - 1}\Big[\int_0^\infty \frac{f(x)}{\max\{x^\lambda, y^\lambda\}}dx\Big]^p dy < \Big(\frac{4}{\lambda}\Big)^p \int_0^\infty x^{p(1-\frac{\lambda}{2})-1} f^p(x)dx; \quad (3.9)$$

(b) *for* $\lambda = 1$, *we have*

$$\int_0^\infty y^{p\psi_q - 1}\Big[\int_0^\infty \frac{f(x)}{\max\{x, y\}}dx\Big]^p dy < \Big(\frac{1}{\phi_p\psi_q}\Big)^p \int_0^\infty x^{p\psi_q-1} f^p(x)dx. \quad (3.10)$$

Proof. Setting a real function $g(y)$ as

$$g(y) = y^{p\psi_q - 1}\Big[\int_0^\infty \frac{f(x)}{\max\{x^\lambda, y^\lambda\}}dx\Big]^{p-1}, \quad y \in (0, \infty),$$

then by (3.1), we find

$$\begin{aligned}
&\Big[\int_0^\infty y^{q(1-\psi_q)-1} g^q(y)dy\Big]^p \\
&= \Big\{\int_0^\infty y^{p\psi_q - 1}\Big[\int_0^\infty \frac{f(x)}{\max\{x^\lambda, y^\lambda\}}dx\Big]^p dy\Big\}^p \\
&= \Big[\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x^\lambda, y^\lambda\}}\Big]^p \\
&\leq \Big(\frac{\lambda}{\phi_p\psi_q}\Big)^p \Big\{\int_0^\infty x^{p(1-\phi_p)-1} f^p(x)dx\Big\}\Big\{\int_0^\infty x^{q(1-\psi_q)-1} g^q(x)dx\Big\}^{p-1}. \quad (3.11)
\end{aligned}$$

Hence we obtain

$$0 < \int_0^\infty y^{q(1-\psi_q)-1} g^q(y)dy \leq \Big(\frac{\lambda}{\phi_p\psi_q}\Big)^p \int_0^\infty x^{p(1-\phi_p)-1} f^p(x)dx < \infty. \quad (3.12)$$

By (3.1), both (3.11) and (3.12) take the form of strict inequality, and we have (3.8).

On the other hand, suppose that (3.8) is valid. By Hölder's inequality, we find

$$\begin{aligned}&\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{\max\{x^\lambda, y^\lambda\}}dxdy\\&=\int_0^\infty \Big[y^{\psi_q-\frac{1}{p}}\int_0^\infty \frac{f(x)}{\max\{x^\lambda, y^\lambda\}}dx\Big]\Big[y^{-\psi_q+\frac{1}{p}}g(y)\Big]dy\\&\leq \Big\{\int_0^\infty y^{p\psi_q-1}\Big[\int_0^\infty \frac{f(x)}{\max\{x^\lambda, y^\lambda\}}dx\Big]^p dy\Big\}^{\frac{1}{p}}\Big\{\int_0^\infty y^{q(1-\psi_q)-1}g^q(y)dy\Big\}^{\frac{1}{q}}. \quad (3.13)\end{aligned}$$

Then by (3.8), we have (3.1). Hence (3.1) and (3.8) are equivalent.

If the constat $(\frac{\lambda}{\phi_p\psi_q})^p$ in (3.8) is not the best possible, by using (3.13), we may get a contradiction that the constant factor in (3.1) is not the best possible. Thus we complete the proof of the theorem.

For $\phi_p = \frac{\lambda}{p}$, $\psi_q = \frac{\lambda}{q}$ by (3.1) and (3.8), we have

Corollary 3.3. *If $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $\lambda > 0$, $f, g \geq 0$ such that $0 < \int_0^\infty x^{p-\lambda-1}f^p(x) < \infty$, and $0 < \int_0^\infty x^{q-\lambda-1}g^q(x)dx < \infty$, then we have*

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{\max\{x^\lambda, y^\lambda\}}dxdy < \frac{pq}{\lambda}\Big\{\int_0^\infty x^{p-\lambda-1}f^p(x)dx\Big\}^{\frac{1}{p}}\Big\{\int_0^\infty x^{q-\lambda-1}g^q(x)dx\Big\}^{\frac{1}{q}}; \quad (3.14)$$

$$\int_0^\infty y^{(p-1)\lambda-1}\Big[\int_0^\infty \frac{f(x)}{\max\{x^\lambda, y^\lambda\}}dx\Big]^p dy < \Big(\frac{pq}{\lambda}\Big)^p\int_0^\infty x^{p-\lambda-1}f^p(x)dx, \quad (3.15)$$

where the constant factors $\frac{pq}{\lambda}$ and $(\frac{pq}{\lambda})^p$ are the best possible. In particular, for $\lambda = 1$, we have

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{\max\{x, y\}}dxdy < pq\Big\{\int_0^\infty x^{p-2}f^p(x)dx\Big\}^{\frac{1}{p}}\Big\{\int_0^\infty x^{q-2}g^q(x)dx\Big\}^{\frac{1}{q}}; \quad (3.16)$$

$$\int_0^\infty y^{p-2}\Big[\int_0^\infty \frac{f(x)}{\max\{x, y\}}dx\Big]^p dy < (pq)^p\int_0^\infty x^{p-2}f^p(x)dx. \quad (3.17)$$

For $\phi_p = \frac{\lambda}{q}$, $\psi_q = \frac{\lambda}{p}$, by (3.1) and (3.8), we have

Corollary 3.4. *If $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $\lambda > 0$, $f, g \geq 0$ such that $0 <$*

$\int_0^\infty x^{(p-1)(1-\lambda)} f^p(x) < \infty$, *and* $0 < \int_0^\infty x^{(q-1)(1-\lambda)} g^q(x)dx < \infty$, *then we have*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x^\lambda, y^\lambda\}} dxdy < \frac{pq}{\lambda} \Big\{ \int_0^\infty x^{(p-1)(1-\lambda)} f^p(x)dx \Big\}^{\frac{1}{p}} \Big\{ \int_0^\infty x^{(q-1)(1-\lambda)} g^q(x)dx \Big\}^{\frac{1}{q}}; \quad (3.18)$$

$$\int_0^\infty y^{\lambda-1} \Big[\int_0^\infty \frac{f(x)}{\max\{x^\lambda, y^\lambda\}} dx \Big]^p dy < \Big(\frac{pq}{\lambda}\Big)^p \int_0^\infty x^{(p-1)(1-\lambda)} f^p(x)dx, \quad (3.19)$$

where the constant factors $\frac{pq}{\lambda}$ *and* $(\frac{pq}{\lambda})^p$ *are the best possible. In particular, for* $\lambda = 1$, (3.18) *reduces to* (1.5).

For $p = q = 2$, by (3.3) and (3.10), we have

Corollary 3.5. *If* $\phi_p,\ \psi_q > 0$, $\phi_p + \psi_q = 1$, $f, g \geq 0$ *such that* $0 < \int_0^\infty x^{2\psi_q - 1} f^2(x) < \infty$, *and* $0 < \int_0^\infty x^{2\phi_p - 1} g^2(x)dx < \infty$, *then we have*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x, y\}} dxdy < \frac{\lambda}{\phi_p \psi_q} \Big\{ \int_0^\infty x^{2\phi_q - 1} f^2(x)dx \Big\}^{\frac{1}{2}} \Big\{ \int_0^\infty x^{2\phi_p - 1} g^2(x)dx \Big\}^{\frac{1}{2}}; \quad (3.20)$$

$$\int_0^\infty y^{2\psi_q - 1} \Big[\int_0^\infty \frac{f(x)}{\max\{x, y\}} dx \Big]^2 dy < \Big(\frac{\lambda}{\phi_p \psi_q}\Big)^2 \int_0^\infty x^{2\psi_q - 1} f^p(x)dx, \quad (3.21)$$

where the constant factors rs *and* $(\frac{\lambda}{\phi_p \psi_q})^2$ *are the best possible.*

Remark.

(a) For $\phi_p = \frac{q+\lambda+2}{q}$, $\psi_q = \frac{p+\lambda+2}{p}$, $(0 < \lambda < 2 - \min\{p, q\})$ inequality (3.1) reduces to (1.7) and for $\phi_p = \frac{p+\lambda+2}{p}$, $\psi_q = \frac{q+\lambda+2}{q}$, $(0 < \lambda < 2 - \min\{p, q\})$ inequality (3.1) reduces to (1.8).

(b) Inequality (3.1) is a new extension of (1.7), (1.8), (3.14) and (3.18).

(c) Inequality (3.20) is an extension of (1.6) with two parameters (r, s).

(d) It is interesting that inequalities (1.7), (1.8), (3.2), (3.14) and (3.18) are different, although they are with the same parameters and posses the best constant factor.

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References

[1] G. H. Hardy, J. E. Littlewood and G. Polya, Inequalities, Cambridge University Press, Cambridge, UK, 1952.

[2] Leping He, Jiangming Yu and Minzhe Gao, *An extension of Hilbert's integral inequality*, J. Shaoguan Univ, (Natural Science), 23:3(2002), 25-30.

[3] Yong Hong, *All-sided generalization about Hardy-Hilbert's integral inequalities*, Acta Math, Sinica, 44:4(2001), 619-626.

[4] Mario Krnić and Josip Pečarić, *General Hilbert's and Hardy's inequalities*, Mathematical Inequalities Applications, 8:1(2005), 29-51.

[5] Jichang Kuang, *On new extensions of Hilbert's integral inequality*, J. Math. Anal. Appl., 235(1999), 608-614.

[6] Jichabg Kuang, Applied Inequalities, Shandong Science and Technology Press, Jiinan, 2004.

[7] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, Inequalities Involving Functions and their Integrals and Derivatives, Kluwer Academic Publishers, Boston, 1991.

[8] Bicheng Yang, *On an extension of Hardy-Hilbert's inequality*, Chinese Annals of Math., 23A:4(2002), 247-254.

[9] Bicheng Yang, *On an extension of Hilbert's inequality with some parameters*, The Australian of J. Math. Anal. Appl., 1:1, Article 11(2004), 1-8.

[10] Bicheng Yang, *On a generalization of the Hilbert's type inequality and its applications*, Chinese Journal of Engineering Mathematics, 21:5(2004).

[11] Bicheng Yang, *On best extensions of Hardy-Hilbert's inequality with two parameters*, Journal of Inequalities in Pure and Applied Mathematics, 6:3, Article 8(2005).

[12] Bicheng Yang, *Best generalization of Hilbert's type of inequality*, Journal of JiLin University (Science Edition), 42:1(2005).

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