LOCAL AND GLOBAL UNIQUENESS THEOREMS ON FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS VIA BIHARI’S AND GRONWALL’S INEQUALITIES

BY

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Abstract. Local and global uniqueness theorems of solutions of the fractional integro-differential equations

\[ D_α^x(t) = f(t, x(t)) + \int_{t_0}^t K(t, s, x(s))ds, \quad 0 < α \leq 1, \]

have been obtained. Our method is an application of Bihari’s and Gronwall’s inequalities.

1. Introduction

Consider the fractional integro-differential equations of the type

\[ D_α^x(t) = f(t, x(t)) + \int_{t_0}^t K(t, s, x(s))ds, \quad α \in \mathbb{R}, \quad 0 < α \leq 1, \]

\[ x(t_0) = x_0, \]

where \( \mathbb{R} \) is the set of real numbers, \( J = [t_0, t_0+a], \) \( a > 0, \) \( f \in C[J \times \mathbb{R}^n, \mathbb{R}^n], \) and \( K \in C[J \times J \times \mathbb{R}^n, \mathbb{R}^n], \) where \( \mathbb{R}^n \) denotes the real \( n \)-dimensional Euclidean space, \( x_0 \) is a real positive constant, and \( D_α^x \) denotes the Caputo fractional derivative operator. The general response expression contains a parameter describing the order of the fractional derivative that can be varied to obtain various responses.
Fractional differential equations have gained importance and popularity during the past three decades or so, due to mainly its demonstrated applications in numerous seemingly diverse fields of science and engineering. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. The advantages of fractional derivatives become apparent in modelling mechanical and electrical properties of real materials, as well as in the description of rheological properties of rocks, and in many other fields (see for details [7, 17]). Among the recent applications we mention areas like the modeling of induction machines [2] and diffusion phenomenon [19].

In recent years, there has been an interest in the study of fractional integro-fractional differential equations. In [9], we used Schauder’s fixed-point theorem to obtain local existence, and Tychonov’s fixed-point theorem to obtain global existence of solutions of (1.1). In [11], we used the successive approximations method and Arzela-Ascoli lemma to obtain existence and uniqueness of solution of equations (1.1). The existence of extremal (maximal and minimal) solutions of the integro-fractional differential equations (1.1) using comparison principle and Ascoli lemma have been investigated in [12]. While in [14], we obtained an asymptotically stable solutions of (1.1). Finally, in [13] we proved some important results concerning with the corresponding inequalities of (1.1).

On the other hand equation (1.1) is a generalization of the fractional differential equations:

\[ D_{t}^{\alpha} x(t) = f(t, x(t)), \quad \alpha \in \mathbb{R}, \quad 0 < \alpha \leq 1, \]

\[ x(t_0) = x_0. \]  

(1.2)

The existence and uniqueness of the solution of (1.2), in additional to some analytical properties and important inequalities, are investigated in [4, 10, 15].

In this paper, we shall use Bihari’s inequality to obtain local uniqueness and Gronwall’s inequality to obtain global uniqueness of solution of the initial value problem (1.1). The results obtained in this paper may be considered as a generalization of the results obtained in [10].
2. Preliminaries and Notations

For the concept of fractional derivative we will adopt Caputo’s definition which is a modification of the Riemann-Liouville definition and has the advantage of dealing properly with initial value problems in which the initial conditions are given in terms of the field variables and their integer order which is the case in most physical processes.

Definition 2.1. A real function \( f(x), \ x > 0, \) is said to be in the space \( C_{\mu}, \mu \in \mathbb{R} \) if there exists a real number \( p(> \mu) \), such that \( f(x) = x^p f_1(x) \), where \( f_1(x) \in C[0, \infty) \), and it is said to be in the space \( C_{\mu}^m \iff f^{(m)} \in C_{\mu}, m \in \mathbb{N} \).

Definition 2.2. The Riemann-Liouville fractional integral operator of order \( \alpha \geq 0 \), of a function \( f \in C_{\mu}, \mu \geq -1 \), is defined as
\[
J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) \, dt, \quad \alpha > 0, \ x > 0,
\]
where \( \Gamma(\alpha) \) is the gamma function.

Properties of the operator \( J^\alpha \) can be found in [6, 8, 16] we mention only the following:

For \( f \in C_{\mu}, \mu \geq -1, \ \alpha, \beta \geq 0 \) and \( \gamma > -1 \):

1. \( J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x) \),
2. \( J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x) \),
3. \( J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma} \).

The Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator \( D^\alpha_\ast \) proposed by M. Caputo in his work on the theory of viscoelasticity [3].

Definition 2.3. The fractional derivative of \( f(x) \) in the Caputo sense is defined as
\[
D^\alpha_\ast f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t) \, dt,
\]
for \( m-1 < \alpha \leq m, \ m \in \mathbb{N}, \ x > 0, \ f \in C_{\mu}^m \).
Also, we need here two of its basic properties.

**Lemma 2.1.** If $m - 1 < \alpha \leq m$, $m \in N$ and $f \in C^m_{\mu}$, $\mu \geq -1$, then

$$D_\alpha^\alpha J_\alpha f(x) = f(x),$$

and,

$$J_\alpha^\alpha D_\alpha^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0.$$

We also need the following results for completeness.

**Lemma 2.2.** The initial value problem (1.1) is equivalent to the nonlinear integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(t, x(s)) ds + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \int_{s}^{t} K(\sigma, s, x(s)) d\sigma ds,$$

(2.3)

where $0 \leq t_0 < t \leq t_0 + a$. In other words, every solution of the integral equation (2.3) is also a solution of our original initial value problem (1.1), and vice versa.

**Proof.** It can be proved easily by applying the integral operator (2.1) to both sides of (1.1), and using some classical results from fractional calculus in [1, 5] to get (2.3).

**Theorem 2.3.** (Bihari’s inequality) Let $g$ be a monotone continuous function in an interval $I$, containing a point $u_0$, which vanishes nowhere in $I$. Let $u$ and $k$ be continuous functions in an interval $J = [t_0, t_0 + c]$ such that $u(J) \subset I$, and suppose that $k$ is of fixed sign in $J$. Let $a \in I$. Suppose that

$$u(t) \leq a + \int_{t_0}^{t} k(s) g(u(s)) ds, \quad t \in J.$$

Then

$$u(t) \leq G^{-1} \left[ G(a) + \int_{t_0}^{t} k(s) ds \right], \quad t \in J,$$

where $G(u)$ is a primitive of $\frac{1}{g(x)}$, i.e. $G(u) = \int_{u_0}^{u} \frac{dx}{g(x)}$, $u \in I$.

**Proof.** The proof of the above theorem can be found in [18].
**Theorem 2.4.** (Gronwall’s inequality) Let \( a(t) \), \( b(t) \) and \( u(t) \) be continuous functions in \( J = [t_0, t_0 + c] \), and let \( b(t) \) be nonnegative in \( J \). Suppose that

\[
 u(t) \leq a(t) + \int_{t_0}^{t} k(s)u(s)ds, \quad t \in J.
\]

Then

\[
 u(t) \leq a(t) + \int_{t_0}^{t} a(s)b(s)\exp\left[\int_{s}^{t} b(\tau)d\tau\right]ds, \quad t \in J.
\]

**Proof.** The proof of the above theorem can be found in [18].

### 3. The Main Theorems

In this section, we shall prove the main theorems, we begin by proving a local uniqueness result by applying Bihari’s inequality.

**Theorem 3.1.** (Local uniqueness theorem) The initial value problem (1.1) has a unique solution on the interval \( t_0 < t \leq t_0 + a \), if the functions \( f \) and \( K \) are continuous in the region

\[
 0 < t_0 < t \leq t_0 + a, \quad |x - x_0| \leq b,
\]

and such that

\[
 |f(t, x) - f(t, y)| \leq \phi(|x - y|), \quad (3.1)
\]

\[
 \int_{s}^{t} |K(\sigma, s, x(s)) - K(\sigma, s, y(s))| d\sigma \leq M\phi(|x - y|), \quad (3.2)
\]

where \( M \) is a positive constant and \( \phi(u) \) is a continuous non-decreasing function on \( 0 < u \leq A \), with \( \phi(0) = 0 \) and

\[
 \int_{0}^{A} \frac{du}{\phi(u)} = +\infty. \quad (3.3)
\]

**Proof.** Assume that there exists two solutions \( x(t) \) and \( y(t) \) of (1.1), both
defined in a neighbourhood at the right of $t_0$. We have

$$
x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(t, x(s)) ds \]

$$

$$
y(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(t, y(s)) ds \]

which lead easily to

$$
| x(t) - y(t) | \leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} | f(t, x(s)) - f(t, y(s)) | ds \]

$$

$$
+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \int_{s}^{t} | K(\sigma, s, x(\sigma)) - K(\sigma, s, y(\sigma)) | d\sigma ds. \]

$$

It follows from (3.1) and (3.2) that

$$
| x(t) - y(t) | \leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \phi(| x(s) - y(s) |) ds \]

$$

$$
+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} M \phi(| x(s) - y(s) |) ds. \]

$$

Thus

$$
| x(t) - y(t) | < \varepsilon + \frac{1 + M}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \phi(| x(s) - y(s) |) ds, \]

$$

where $\varepsilon > 0$. We can now apply Bihari’s inequality to obtain

$$
| x(t) - y(t) | < \Phi^{-1} \left[ \Phi(\varepsilon) + \frac{(1+M)(t-t_0)^{\alpha}}{\alpha \Gamma(\alpha)} \right], \quad t \in [t_0, t_0 + a], \quad (3.4) \]

$$

where $\Phi(u)$ is a primitive of the function $\frac{1}{\phi(u)}$, and $\Phi^{-1}$ denotes the inverse of $\Phi$.

We shall prove that the right-hand side of (3.4) tends toward zero as $\varepsilon \to 0$. Since $| x(t) - y(t) |$ is independent of $\varepsilon$, it follows that $x(t) \equiv y(t)$, which we need. Let us remark that condition (3.3) implies that $\Phi(\varepsilon) \to -\infty$ for $\varepsilon \to 0$, no matter how we choose the primitive of $\frac{1}{\phi(u)}$. Thus $\Phi^{-1}(u) \to 0$ as $u \to -\infty$. 
Consequently, when $\varepsilon \to 0$ in the inequality (3.4), the right-hand side tends towards zero (for all finite $t$).

Therefore, $x(t) = y(t)$, for $t \in [t_0, t_0 + a]$, and the theorem is proved.

We shall next discuss a global uniqueness result for the initial value problem (1.1) using Gronwall’s inequality.

**Theorem 3.2.** (Global uniqueness theorem) Assume that

(i) $f$ and $K$ are continuous functions in the region

$$D = \{(t, x) : 0 < t_0 < t \leq t_0 + a, \ |x - x_0| \leq b\} \subset \Omega,$$

where $\Omega$ is an open $(t, x)$-set in $\mathbb{R}^{n+1}$.

(ii) $f$ and $K$ satisfy a local Lipschitz condition, with respect to $x$,

$$|f(t, x) - f(t, y)| \leq L|x - y|,$$

and

$$\int_s^t |K(\sigma, s, x(s)) - K(\sigma, s, y(s))| \, d\sigma \leq L^2|x - y|,$$

for some positive constant $L$.

(iii) $x(t)$ and $\tilde{x}(t)$ are solutions of (1.1), such that their intervals of definition have common points and $x(t_0) = \tilde{x}(t_0)$, in such a point.

Then $x(t) = \tilde{x}(t)$ on the common interval of definition.

**Proof.** Assume that $(t_1, t_2)$ is the interval where both solutions are defined. Then $t_0 \in (t_1, t_2)$. It suffices to prove that $x(t) = \tilde{x}(t)$ for $t_0 \leq t < t_2$.

Consider now a number $T$, such that $t_0 < T < t_2$. It will be fixed in the first step of the proof, but we want to point out that it can be chosen arbitrarily close to $t_2$. Let $H \subset \Omega$ be a compact set such that

$$(t, x(t)), \ (t, \tilde{x}(t)) \in H, \text{ for } t \in [t_0, T].$$

The existence of the set $H$, with the preceding property, is the consequence of the fact that both sets $\{(t, x(t)); \ t \in [t_0, T]\}$ and $\{(t, \tilde{x}(t)); \ t \in [t_0, T]\}$ are compact, which follows easily from the continuity of $x(t)$ and $\tilde{x}(t)$.

Denote by $x_0$ the common value of the solutions $x(t)$ and $\tilde{x}(t)$ at $t = t_0$. 
For $t \in [t_0, T]$ we shall have

\[
x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(t, x(s)) ds \\
+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \int_{s}^{t} K(\sigma, s, x(s)) d\sigma ds,
\]

\[
\tilde{x}(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(t, \tilde{x}(s)) ds \\
+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \int_{s}^{t} K(\sigma, s, \tilde{x}(s)) d\sigma ds,
\]

from which we get

\[
| x(t) - \tilde{x}(t) | \leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} | f(t, x(s)) - f(t, \tilde{x}(s)) | ds \\
+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \int_{s}^{t} | K(\sigma, s, x(s)) - K(\sigma, s, \tilde{x}(s)) | d\sigma ds.
\]

\[
| x(t) - \tilde{x}(t) | \leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} L | x(s) - \tilde{x}(s) | ds \\
+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} L^2 | x(s) - \tilde{x}(s) | ds.
\]

Thus

\[
| x(t) - \tilde{x}(t) | \leq \varepsilon + \frac{L + L^2}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} | x(s) - \tilde{x}(s) | ds,
\]

(3.5)

for any $\varepsilon > 0$ and $t \in [t_0, T]$.

Inequality (3.5) is of Gronwall type, therefore, the application of Gronwall’s Theorem yields

\[
| x(t) - \tilde{x}(t) | \leq \varepsilon + \frac{\varepsilon \alpha (L + L^2)}{\alpha \Gamma(\alpha)} \exp \left[ \frac{(t-t_0)^\alpha}{\alpha} \right], \ t \in [t_0, T].
\]

(3.6)

Since $\varepsilon$ is arbitrary, inequality (3.6) implies that $x(t) = \tilde{x}(t)$ on $[t_0, T]$. On the other hand, $T$ can be chosen arbitrarily close to $t_2$, which proves that $x(t) = \tilde{x}(t)$ on $[t_0, t_2]$.

Hence the theorem is proved.
References


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