SUPER EFFICIENCY IN VECTOR OPTIMIZATION PROBLEMS WITH IC-CONE-CONVEXLIKE SET-VALUED MAPS

BY

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Abstract. This paper gives an useful property of ic-cone-convexlike set-valued maps. By using this property as the main tool, the scalarization theorem and optimality conditions for super efficient solutions of a vector optimization problem with the ic-cone-convexlike objectives and constraints in normed linear spaces are proposed. In addition, it shows that a super efficient solution can be expressed in terms of saddle point.

1. Introduction

In recent years, much attention has been paid to generalized convexity in vector optimization with set-valued mappings. This is because this property can replace the convexity requirement in proving many interesting results in set-valued optimization theory and related topics.

Let $Q$ be an arbitrary set, $D$ and $E$ be convex cones of normed linear spaces $Y$ and $Z$. Let $F : Q \to 2^Y$ and $G : Q \to 2^Z$ be two set-valued maps associating to any point $x \in Q$ the nonempty set $F(x)$ and $G(x)$, respectively. In this work, we are interested in super efficient solutions of the following vector optimization

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problem (SOP):

\[
\begin{align*}
\text{min} & \ F(x) \\
\text{s.t.} & \ x \in \Omega := \{ x' \in Q : G(x') \cap (-E) \neq \emptyset \}
\end{align*}
\]

where \( \emptyset \) is the empty set. For problem (SOP), many investigation results for different efficiencies have been obtained under generalized convexity: cone-convexlikeness ([4, 9]), cone-subconvexlikeness ([4-7, 9-10]), generalized cone-subconvexlikeness ([3, 11]) and near cone-subconvexlikeness ([11-13]). Among them, the near cone-subconvexlikeness is the most general notion. Recently, Sach ([12]) introduced a new notion of generalized convexity for set-valued maps, called ic-cone-convexlikeness. This kind of generalized convexity is broader than near cone-subconvexlikeness. Using it as the main tool, Sach derive an alternative theorem and some results for efficient, weakly efficient, and Benson properly efficient solutions of the set-valued optimization problems ([12]). Borwein and Zhuang introduced the concept of super efficiency in normed linear spaces ([1]). Super efficiency refines the notion of efficiency and other kinds of proper efficiency. Borwein and Zhuang provided concise scalar characterizations and duality results when underlying decision problem is convex. Xu and Liu gives some characterizations of the super efficient solutions in terms of scalarization and Lagrangian multiplier by means of the near cone-subconvexlikeness in normed linear space ([13]). In this paper, we obtain a scalarization theorem and optimality conditions for super efficient solutions by means of the notion of ic-cone-convexlikeness for problem (SOP). In addition, we show that a super efficient solution can be expressed in terms of saddle point.

Throughout this paper, it is assumed that \( X \) is an arbitrary set and that \( Y \) and \( Z \) are normed linear spaces with topological duals \( Y^* \) and \( Z^* \), respectively. The origin of \( Y \) is denoted by \( 0_Y \). A set \( A \subset Y \) is a cone if \( \lambda A \subset A \), \( \forall \lambda > 0 \). A cone \( A \) is pointed if \( a \in A \cap (-A) \Rightarrow a = 0_Y \). For a cone \( A \subset Y \), we set

\[
A^+ = \{ y^* \in Y^* : \langle y^*, a \rangle \geq 0, \ \forall a \in A \},
\]

where \( \langle \cdot, \cdot \rangle \) denotes the canonical bilinear form between \( Y \) and \( Y^* \). For a set \( A \subset Y \), we write

\[
\text{cone}A = \{ \lambda a : \lambda > 0, \ a \in A \}.
\]
If $D$ is a cone, then it has been known that
\
cone(A + D) = coneA + D. \tag{1.1}
\
The closure and interior of a set $A$ are denoted by $clA$ and $intA$. A convex subset $B$ of a cone $A$ is a base of $A$ if $0_Y \notin clB$ and $A \setminus \{0_Y\} = coneB$.

The following lemma is known in convex analysis.

**Lemma 1.** ([12]) Let $A \subset Y$ be a convex set. Then $clA$ is convex. In addition, if $intA \neq \emptyset$, then $intA$ is convex, $clA = cl(intA)$, and $intA = int(clA)$.

**Lemma 2.** ([12]) Let $A \subset Y$ be an arbitrary subset and $D \subset Y$ be a convex cone with nonempty interior. then
\
cl(A + D) = cl(A + intD), \tag{1.2}

int[cl(A + D)] = A + intD = int(A + D). \tag{1.3}

### 2. A Property of ic-Cone-Convexlike Set-Valued Maps

Let $A \subset Y$ be a nonempty subset and $D \subset Y$ be a nonempty convex cone.

**Definition 1.** ([12]) The set $A$ is called int-convex (shortly, i-convex) if $intA$ is convex and if $A \subset cl(intA)$.

**Definition 2.** ([12]) The set $A$ is called intcone-convex (shortly, ic-convex) if $coneA$ is i-convex.

**Definition 3.** ([12]) The set $A$ is called ic-$D$-convex if $A + D$ is ic-convex.

Now, for a set-valued map $F : X \to 2^Y$, denote by $domF$ and $imF$ the domain and the image of $F$,
\
$domF = \{x \in X : F(x) \neq \emptyset\}$,

$imF = F(X) = \bigcup_{x \in Q} F(x)$.

**Definition 4.** ([12]) The map $F$ is called ic-$D$-convexlike if $imF$ is ic-$D$-convex.
Lemma 3. ([12]) Let $\text{int}[\text{cone}(\text{im}F + D)] \neq \emptyset$. The map $F$ is ic-$D$-convexlike if and only if $\text{cl}[\text{cone}(\text{im}F + D)]$ is convex and $\text{int}[\text{cl}[\text{cone}(\text{im}F + D)]] = \text{int}[\text{cone}(\text{im}F + D)]$.

Let $K \subset Y$ be a closed pointed convex cone with $\text{int}K \neq \emptyset$ and $f$ be a vector valued map from $X$ into $Y$.

Definition 5. ([2]) The map $f$ is said to be $K$-subconvexlike if we can find a $\theta \in \text{int}K$ such that, for any $x_1, x_2 \in Q$, any $\lambda \in (0, 1)$, and any $\varepsilon > 0$, there exists $x_3 \in Q$ satisfying
\[
\varepsilon \theta + \lambda f(x_1) + (1 - \lambda)f(x_2) - f(x_3) \in K.
\]

Lemma 4. ([2]) The map $f$ is K-subconvexlike if and only if $\text{im}f + \text{int}K$ is convex.

The following theorem is very useful. We shall use it as the main tool to discuss the super efficiency of the problem (SOP).

Theorem 1. Let $K_1, K_2$ be closed pointed convex cones with nonempty interior and $K_1 \subset K_2$. Let $\text{int}[\text{cone}(\text{im}F + K_1)] \neq \emptyset$. If $F$ is ic-$K_1$-convexlike, then $F$ is ic-$K_2$-convexlike.

Proof. Firstly, by Lemma 3, $\text{cl}[\text{cone}(\text{im}F + K_1)]$ is convex and $\text{int}[\text{cl}[\text{cone}(\text{im}F + K_1)]] = \text{int}[\text{cone}(\text{im}F + K_1)]$.

This implies that $\text{int}[\text{cone}(\text{im}F + K_1)]$ is convex. On the other hand, setting $\lambda = \text{im}F$ and making use of Lemma 2 and equality $1.1$, we get
\[
\text{int}[\text{cone}(\text{im}F) + K_1] = \text{cone}(\text{im}F) + \text{int}K_1
\]
is convex. Let $f = I : \text{cone}(\text{im}F) \rightarrow Y$ be identity mapping, then
\[
\text{im}f + \text{int}K_1 = \text{cone}(\text{im}F) + \text{int}K_1.
\]
Thus, $\text{im}f + \text{int}K_1$ is convex. By lemma 4, $f$ is $K_1$-subconvexlike. So we can find a $\theta \in \text{int}K_1$ such that for any $x_1, x_2 \in \text{cone}(\text{im}F)$, any $\lambda \in (0, 1)$, and any $\varepsilon > 0$, there exists $x_3 \in \text{cone}(\text{im}F)$ satisfying $\varepsilon \theta + \lambda f(x_1) + (1 - \lambda)f(x_2) - f(x_3) \in K_1$.

Since $K_1 \subset K_2$, it is obviously that $\theta \in \text{int}K_2$ and $\varepsilon \theta + \lambda f(x_1) + (1 - \lambda)f(x_2) - f(x_3) \in K_2$. 
This yields that \( f \) is \( K_2 \)-subconvexlike. Making use of Lemma 4 again, we obtain that \( \text{im} f + \text{int} K_2 = \text{cone}(\text{im} F) + \text{int} K_2 \) is convex. Setting \( A = \text{im} F \), by Lemma 2 and equality (1.1) again, we can see that

\[
\text{cl}[\text{cone}(\text{im} F) + \text{int} K_2] = \text{cl}[\text{cone}(\text{im} F) + K_2] = \text{cl}[\text{cone}(\text{im} F + K_2)]
\]
is convex, and

\[
\text{int}[\text{cone}(\text{im} F + K_2)] = \text{int}[\text{cone}(\text{im} F) + K_2] = \text{cone}(\text{im} F) + \text{int} K_2 = \text{int}[\text{cl}(\text{cone}(\text{im} F + K_2))]
\]

Using Lemma 3, we prove that \( F \) is ic-\( K_2 \)-convexlike.

**Lemma 5.** ([12]) Let \( F \) be ic-\( D \)-convexlike. Then, either

(a) \( 0_Y \in \text{int}[\text{cone}(\text{im} F + D)] \), or

(b) \( \exists y^* \in D^+ \backslash \{0_Y\} \) s.t. \( \inf_{y \in \text{im} F} \langle y^*, y \rangle \geq 0 \),

but never both.

### 3. Scalarization and Optimality Conditions for Super Efficiency

Throughout this paper, let \( F : X \rightarrow 2^Y \) and \( G : X \rightarrow 2^Z \) be set-valued maps with the same domain \( Q \). Let \( D \subset Y \) and \( E \subset Z \) be closed pointed convex cones with nonempty interior. From now on, we assume that \( D \neq Y \) and \( D \neq 0_Y \). Consider the vector optimization problem (SOP) formulated in Section 1. We assume that \( x_0 \in \Omega, y_0 \in F(x_0), \) and \( z_0 \in G(x_0) \cap (-E) \).

**Definition 6.** ([12]) A point \((x_0, y_0)\) is a weakly efficient solution of (SOP) with respect to \( D \) (written as \( y_0 \in \text{WE}[F(\Omega), D] \)) if \( [F(\Omega) - y_0] \cap (-\text{int} D) = \emptyset \).

**Definition 7.** ([10]) A point \((x_0, y_0)\) is a super efficient of (SOP) with respect to \( D \) (written as \( y_0 \in \text{SE}[F(\Omega), D] \)) if there exists a real number \( k > 0 \) such that

\[
\text{cl}(\text{cone}[F(\Omega) - y_0]) \cap (U - D) \subset kU,
\]

where \( U \) is the closed unit ball of \( Y \).
Let $B$ be a closed bounded base of $D$, and let $D_\alpha = cl\{cone(B + \alpha U)\}$, where $\alpha \in (0, \delta)$ and $\delta = \inf \{||b|| : b \in B\}$. It has been proved that for each $\alpha \in (0, \delta)$, $D_\alpha$ is a convex cone and $D \subset D_\alpha$.

**Lemma 6.** ([1]) For any $\alpha \in (0, \delta)$, $D_\alpha \setminus \{0_Y^*\} \subset int D^+, D \setminus \{0_Y\} \subset int D_\alpha$.

**Lemma 7.** ([1]) Let $B$ be a closed bounded base of $D$. If $(x_0, y_0)$ is a super efficient solution of (SOP) with respect to $D$, then there exists $\alpha \in (0, \delta)$ such that $(x_0, y_0)$ is a weakly efficient solution of (SOP) with respect to $D_\alpha$.

Now, consider a scalar minimization problem $(P_\mu)$ for problem (SOP):

$$
(P_\mu) \begin{cases} 
\min \langle \mu, F(x) \rangle \\
\text{s.t. } x \in \Omega 
\end{cases}
$$

where $\mu \in Y^* \setminus \{0_Y^*\}$.

**Definition 8.** A point $(x_0, y_0)$ is called an optimal solution of $(P_\mu)$, if $\langle \mu, y \rangle \geq \langle \mu, y_0 \rangle, \forall y \in F(\Omega)$.

**Theorem 2.** Let $int D \neq \emptyset$ and $B$ be a closed bounded base of $D$. Let $(x_0, y_0)$ be a super efficient solution of (SOP) with respect to $D$. Suppose that $int[cone(im(F - y_0) + D)] \neq \emptyset$. If $F - y_0$ is ic-$D$-convexlike, then there exists $\bar{\mu} \in int D^+$ such that $(x_0, y_0)$ is an optimal solution of $(P_{\bar{\mu}})$.

**Proof.** We claim that there exists $\bar{\alpha} \in (0, \delta)$ such that

$$0_Y \notin int[cone(im(F - y_0) + D_{\bar{\alpha}})]. \quad (3.1)$$

Indeed, otherwise for all $\alpha' \in (0, \delta)$ we have

$$0_Y \in int[cone(im(F - y_0) + D_{\alpha'})].$$

Making use of Lemma 2 and observing that $D_{\alpha'}$ is a convex cone with nonempty interior, we see that

$$0_Y \in int[cone(im(F - y_0) + D_{\alpha'})] \iff 0_Y \in int[cone(im(F - y_0)) + D_{\alpha'}] \iff 0_Y \in cone(im(F - y_0)) + int D_{\alpha'} \iff im(F - y_0) \cap (-int D_{\alpha'}) \neq \emptyset.$$
Hence, for each \( \alpha' \in (0, \delta) \), we obtain that
\[
\text{im}(F - y_0) \cap (-\text{int}D_{\alpha'}) \neq \emptyset. \tag{3.2}
\]
Since \((x_0, y_0)\) is a super efficient solution of (SOP) with respect to \(D\), by Lemma 6, there exists \( \bar{\alpha} \in (0, \delta) \) such that \((x_0, y_0)\) is a weakly efficient solution of (SOP) with respect to \(D_{\bar{\alpha}}\). That is
\[
\text{im}(F - y_0) \cap (-\text{int}D_{\bar{\alpha}}) = \emptyset.
\]
This contracts to (3.2). Thus, we have shown that (3.1) holds. Observing that \(D \subset D_{\bar{\alpha}}\) and applying Theorem 1 and Lemma 5, we can find \(\bar{\mu} \in D_{\bar{\alpha}}^+ \setminus \{0_Y^*\}\) such that
\[
\langle \bar{\mu}, y \rangle \geq \langle \bar{\mu}, y_0 \rangle, \quad \forall y \in \text{im}F.
\]
Furthermore, by Lemma 6, we obtain that \(D_{\bar{\alpha}}^+ \setminus \{0_Y^*\} \subset \text{int}D^+\). Therefore, there exists \(\bar{\mu} \in \text{int}D^+\) such that \((x_0, y_0)\) is optimal solution of \((P_{\bar{\mu}})\). This completes the proof.

For each \(\beta \in [0, 1)\), let us consider a set-valued map \(H_\beta : X \rightarrow 2^{Y \times Z}\) whose domain is the set \(Q\),
\[
H_\beta(x) = (F(x) - y_0) \times (G(x) - \beta z_0), \quad x \in Q.
\]
Let \(K = D \times E\). From now on, we make the following assumption.

**Assumption (A).** ([12]) There exists \(\beta \in [0, 1)\) such that \(H_\beta\) is ic-\(K\)-convexlike.

**Definition 9.** ([12]) It is said that condition (CQ) holds if \(\text{cl}[\text{cone}(\text{im}G + E)] = Z\).

**Lemma 8.** ([12]) Let \(\text{int}D \neq \emptyset\). Let Assumption (A) be satisfied. Let \((x_0, y_0)\) be a weakly efficient solution of problem (SOP). If the condition (CQ) holds, then there exists \((y_0^*, z_0^*) \in D^+ \times E^+\) such that
\[
\langle y_0^*, y \rangle + \langle z_0^*, z \rangle \geq \langle y_0^*, y_0 \rangle, \quad \forall (y, z) \in \text{im}(F \times G) \tag{3.2}
\]
\[
\langle z_0^*, z'_0 \rangle = 0, \quad \forall z'_0 \in G(x_0) \cap (-\text{cl}E), \tag{3.3}
\]
where \(y_0^* \neq 0_{Y^*}\).
Theorem 3. Let $\text{int} \, D \neq \emptyset$. Let Assumption (A) be satisfied. Let $(x_0, y_0)$ be a super efficient solution of problem (SOP) with respect to $D$. Suppose that $\text{int}[\text{cone}(\text{im} H_\beta + D \times E)] \neq \emptyset$. If the condition (CQ) holds, then there exists $(y^*_0, z^*_0) \in \text{int} D^+ \times E^+$ such that conditions (3.3) and (3.4) are fulfilled.

Proof. Since $(x_0, y_0)$ be a super efficient solution of problem (SOP) with respect to $D$, by Lemma 7, there exists $\alpha \in (0, \delta)$ such that $(x_0, y_0)$ is a weakly efficient solution of (SOP) with respect to $D_\alpha$. Observing that $D \subset D_\alpha$ and applying Theorem 1, we obtain that $H_\beta$ is ic-$K_\alpha$-convexlike, where $K_\alpha = D_\alpha \times E$.

By Lemma 8, we can find $(y^*_0, z^*_0) \in D_\alpha^+ \times E^+$ such that conditions (3.3) and (3.4) are fulfilled. On the other hand, since $y^*_0 \in D_\alpha^+ \{0_Y\}$, we get from Lemma 6 that $y^*_0 \in \text{int} D_\alpha^+$ as desired.

4. Super Efficient Solutions and Saddle Points

In this section, we show that a super efficient solution of the problem (SOP) can be expressed in terms of saddle points defined in a suitable sense.

Definition 10. ([11]) Let $y^*_0 \in D^+$. A pair $(x_0, z^*_0) \in Q \times E^+$ is called a saddle point of (SOP) with respect to $y^*_0$ if there exists $(y_0, z_0) \in F(x_0) \times G(x_0)$ such that

$$
\langle y^*_0, y_0 \rangle + \langle z^*_0, z_0 \rangle \leq \langle y^*_0, y \rangle + \langle z^*_0, z \rangle \leq \langle y^*_0, y_0 \rangle + \langle z^*_0, z_0 \rangle,
$$

whenever $x \in Q$, $(y, z) \in F(x) \times G(x)$, $z^* \in E^+$.

Lemma 9. ([11])

(a) Let $x_0 \in \Omega$, $y_0 \in F(x_0)$ and $(y^*_0, z^*_0) \in D^+ \times E^+$ be such that

$$
\forall x \in Q, \quad \forall (y, z) \in F(x) \times G(x), \quad \langle y^*_0, y - y_0 \rangle + \langle z^*_0, z \rangle \geq 0.
$$

Then, $(x_0, z^*_0)$ is a saddle point of (SOP) with respect to $y^*_0$.

(b) Conversely, let $(x_0, z^*_0) \in Q \times E^+$ be a saddle point of (SOP) with respect to $y^*_0 \in D^+$. Then,

(i) $\forall x \in Q, \forall (y, z) \in F(x) \times G(x)$, (4.2) holds where $y_0 \in F(x_0)$ is the point appearing in Definition 10.
\(\forall z_0^* \in G(x_0) \cap (-E), \langle z_0^*, z'_0 \rangle \geq 0.\)

(iii) \(x_0 \in \Omega\) if \(E\) is a closed (convex) cone.

**Theorem 4.** Let \(x_0 \in \Omega, H = \bigcup_{x \in Q} F(x) \times G(x).\)

(I) Let \(y_0^* \in \text{int}D^+\) and \(z_0^* \in E^+\). If \((x_0, z_0^*)\) is a saddle point of \((SOP)\) with respect to \(y_0^*.\) Then

\[
\text{cl}[\text{cone}((y_0, 0_Z) - H - D \times E)] \cap D \times \{0_Z\} = (0_Y, 0_Z),
\]

where \(y_0 \in F(x_0)\) is the point appearing in Definition 10.

(II) Let \(y_0^* \in D^+ \setminus \{0_Y\}\) and \(z_0^* \in E^+.\) If \((x_0, z_0^*)\) is a saddle point of \((SOP)\) with respect to \(y_0^*\). Then

\[
\text{cone}((y_0, 0_Z) - H - D \times E) \cap \text{int}D \times \{0_Z\} = \emptyset,
\]

where \(y_0 \in F(x_0)\) is the point appearing in Definition 10.

**Proof.** We only prove the statement (I), since the proof of statement (II) is similar to that of statement (I). Suppose that there exist \(y_0^* \in \text{int}D^+\) and \(z_0^* \in E^+\) such that \((x_0, z_0^*)\) is a saddle point of \((SOP)\) with respect to \(y_0^*.\) By Lemma 9, there exists \((x_0, y_0) \in V \times F(x_0)\) such that (4.2) holds. If (4.3) is not fulfilled, then we can find \(\bar{y} \in D\) with \(\bar{y} \neq 0_Y\) and

\[
(\bar{y}, 0_Z) \in \text{cl}[\text{cone}((y_0, 0_Z) - H - D \times E)] \cap D \times \{0_Z\}.
\]

Hence,

\[
\langle y_0^*, \bar{y} \rangle + \langle z_0^*, 0_Z \rangle = \langle y_0^*, \bar{y} \rangle > 0.
\]

By the continuity of \(y_0^*\) and \(z_0^*,\) there exists \((y', z') \in \text{cone}((y_0, 0_Z) - H - D \times E)\) such that

\[
\langle y_0^*, y' \rangle + \langle z_0^*, z' \rangle > 0.
\]

So, there exist \(\alpha > 0, x_1 \in Q, y_1 \in F(x_1), z_1 \in G(x_1)\) and \((d_1, e_1) \in D \times E\) such that

\[
\langle y_0^*, \alpha(y_0 - y_1 - d_1) \rangle + \langle z_0^*, \alpha(-z_1 - e_1) \rangle > 0.
\]

Since \(\langle y_0^*, -d_1 \rangle \leq 0\) and \(\langle z_0^*, -e_1 \rangle \leq 0,\) we obtain that

\[
\alpha(\langle y_0^*, y_0 - y_1 \rangle + \langle z_0^*, -z_1 \rangle) > 0.
\]
Thus,
\[ \langle y_0^*, y_1 - y_0 \rangle > \langle z_0^*, z_1 \rangle. \]
This is contradiction to (4.2).

**Lemma 10.** ([13]) Suppose \( D \) has closed bounded base \( B \), and \( y_0^* \in \text{int}D^+ \), \( d_n \in D \), \( n = 1, 2, \ldots \). If \( ||d_n|| \to +\infty \), then \( \langle y_0^*, d_n \rangle \to +\infty \).

**Theorem 5.** Suppose \( D \) has closed bounded base \( B \), and \( E \) is a closed convex cone. If \( (x_0, z_0^*) \in Q \times E^+ \) is a saddle point of (SOP) with respect to \( y_0^* \in \text{int}D^+ \), then for some \( y_0 \in F(x_0) \), \( (x_0, y_0) \) is a super efficient solution of (SOP) with respect to \( D \).

**Proof.** Since \((x_0, z_0^*)\) is a saddle point of (SOP) with respect to \( y_0^* \in \text{int}D^+ \), we obtain by Lemma 9 (b) that \( x_0 \in \Omega \) and for some \( y_0 \in F(x_0) \), (4.2) holds. Assuming that for this \( y_0 \in F(x_0) \), \( (x_0, y_0) \) is not a super efficient solution of (SOP) with respect to \( D \). Thus, for all \( k > 0 \), we see that
\[
\text{cl}(\text{cone}[F(\Omega) - y_0]) \cap (U - D) \nsubseteq kU,
\]
where \( U \) is the closed unit ball of \( Y \). So there exist \( a_n^m > 0, x_n^m \in V, y_n^m \in F(x_n^m), u_n \in U, d_n \in D, m, n = 1, 2, \ldots \), such that
\[
u_n - d_n = \lim_{m \to +\infty} e_n^m(y_n^m - y_0), \quad n = 1, 2, \ldots,
\]
and
\[ ||u_n - d_n|| \to +\infty, \quad n \to +\infty. \]
Since \( u_n \in U \), it is obviously that \( ||d_n|| \to +\infty \). By Lemma 10, it yields that \( \langle y_0^*, d_n \rangle \to +\infty \). So,
\[
\langle y_0^*, u_n - d_n \rangle = \langle y_0^*, u_n \rangle - \langle y_0^*, d_n \rangle \to -\infty, \quad n \to +\infty.
\]
Therefore there exists a positive integrity number \( N \) such that
\[ 0 > \langle y_0^*, u_N - d_N \rangle = \lim_{n \to \infty} a_N^M(\langle y_0^*, y_N^m - y_0 \rangle). \]
Thus, for some positive integer \( M \) we have
\[ a_N^M(\langle y_0^*, y_N^m - y_0 \rangle) < 0, \quad \text{or equivalently} \quad \langle y_0^*, y_N^M - y_0 \rangle < 0. \]
since $a_M^N > 0$. On the other hand, by condition $x_M^N \in \Omega$ there exists $z_N^M \in G(x_M^N) \cap -E$ for which $\langle z_0^*, z_N^M \rangle \leq 0$ since $z_0^* \in E^+$. Therefore,

$$\langle y_0^*, y_M^N - y_0 \rangle + \langle z_0^*, z_N^M \rangle < 0,$$

a contradiction to (4.2). This completes the proof.

The following Theorem 6 is a direct consequence of Theorem 3 and Lemma 9.

**Theorem 6.** Let $x_0 \in \Omega$, $y_0 \in F(x_0)$, $z_0 \in G(x_0) \cap (-E)$ and $\text{int}D \neq \emptyset$. Let Assumption (A) be satisfied and the condition (CQ) hold. Suppose that $\text{int} \text{cone}(\text{im}H_\beta + D \times E) \neq \emptyset$. If $(x_0, y_0)$ is a super efficient solution of problem (SOP) with respect to $D$, then there exists $(y_0^*, z_0^*) \in \text{int}D^+ \times E^+$ such that $(x_0, z_0^*)$ is a saddle point of problem (SOP) with respect to $y_0^*$.

At last, we establish a theorem which shows that a super efficient solution of problem (SOP) is a optimal solution of the scalar minimization problem $(P_\mu)$.

**Theorem 7.** Assuming that $x_0 \in Q$, $y_0^* \in D^+ \backslash \{0\}$ and $\langle x_0, z_0^* \rangle \in Q \times E^+$ is a saddle point of problem (SOP) with respect to $y_0^*$. Then for some $y_0 \in F(x_0)$, $(x_0, y_0)$ is a optimal solution of the problem $(P_\mu)$.

**Proof.** By Lemma 9, $x_0 \in \Omega$, and for some $y_0 \in F(x_0)$ such that (4.2) holds. Suppose that $(x_0, y_0)$ is not an optimal solution of the problem $(P_\mu)$, then there exist $x' \in \Omega$, $y' \in F(x')$ such that $\langle y_0^*, y' - y_0 \rangle < 0$. Since $x' \in \Omega$, there exists $z' \in G(x') \cap (-E)$ such that $\langle z_0^*, z' \rangle \leq 0$. Thus, $\langle y_0^*, y' - y_0 \rangle + \langle z_0^*, z' \rangle < 0$, which is a contradiction to (4.2).

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