Abstract. The purpose of present note is to study the totally umbilical semi-invariant submanifolds of a nearly Kenmotsu manifold. In this note we have discussed the integrability of invariant and anti-invariant distributions and consequently obtained a classification for the totally umbilical semi-invariant submanifold of a nearly Kenmotsu manifold.

1. Introduction

I. Mihai et al. [7] generalized the notion of Kenmotsu manifold. In fact they introduced the structure of $f$-Kenmotsu manifold. Several Authors studied the semi-invariant submanifold of Kenmotsu manifolds and $f$-Kenmotsu manifolds, a generalized version of Kenmotsu manifold, it is therefore worthwhile to study the semi-invariant submanifolds of a nearly Kenmotsu manifold. An almost contact metric manifold $\bar{M}(\phi, \xi, \eta, g)$ satisfying $(\bar{\nabla}_X \phi)X = -\eta(X)\phi X$, is called a nearly Kenmotsu manifold ([6]), where $\bar{\nabla}$ is the Riemannian connection corresponding to the contact metric on $\bar{M}$. Obviously a nearly Kenmotsu structure on an almost contact metric manifold is given by a slightly weaker condition than that of a Kenmotsu manifold. The study of geometry of CR-submanifolds of a Kähler manifold was initiated by A. Bejancu [1]. In particular the totally umbilical and totally geodesic CR-submanifolds of a Kähler manifold have also been extensively studied. Later, Chen [5] classified totally umbilical CR-submanifold of Kähler manifolds. In the present note we obtain a characterization of totally umbilical semi-invariant submanifolds of a nearly Kenmotsu manifold.
2. Preliminaries

Let $\bar{M}$ be a $2n+1$ dimensional nearly Kenmotsu manifold with structure tensors $(\phi, \xi, \eta, g)$, then they satisfy
\begin{align}
\phi^2 X &= -X + \eta(X)\xi, \quad \phi(\xi) = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1, \\
\eta(\phi_X) &= 0, \quad \eta(\xi) = 1,
\end{align}
\tag{2.1}
\forall X, Y \in T\bar{M}.

An $m$-dimensional submanifold $M$ of $\bar{M}$ is said to be a semi-invariant submanifold if there exist a pair of orthogonal distributions $(D, D^\perp)$ satisfying the conditions
(i) $TM = D \oplus D^\perp \oplus \langle \xi \rangle$,
(ii) the distribution $D$ is invariant by $\phi$, i.e., $\phi D_x = D_x$, $\forall x \in M$,
(iii) the distribution $D^\perp$ is anti-invariant i.e., $\phi D^\perp_x \subseteq T^\perp_x M$, $x \in M$,
where $\langle \xi \rangle$ is the distribution spanned by structure vector field $\xi$.

Let $TM$ denote the tangent bundle on $M$. The orthogonal complement of $\phi D^\perp$ in the normal bundle $\gamma(M)$ is an invariant subbundle of $\gamma(M)$ under $\phi$ and denoted by $\mu$. For $U, V \in TM$ and $N \in \gamma(M)$ the transform $\phi U$ and $\phi N$ are decomposed into tangential and normal parts respectively as
\begin{align}
\phi U &= PU + FU, \\
\phi N &= tN + fN.
\end{align}
\tag{2.4}
\tag{2.5}

Here it is easy to observe that $PU \in D$, $FU \in \phi D^\perp$, $tN \in D^\perp$ and $fN \in \mu$. Similarly denoting by $P_U V$ and $Q_U V$ the tangential and normal parts of $(\nabla_U \phi)V$ and making use of equations (2.4), (2.5), the Gauss and Weingarten formulae, the following equation may easily obtained
\begin{align}
P_U V &= (\nabla_U P)V - A_{FU} U - th(U, V), \\
Q_U V &= (\nabla_U F)V + h(U, PV) - fh(U, V),
\end{align}
\tag{2.6}
\tag{2.7}

where the covariant derivatives of $P$ and $F$ are defined by
\begin{align}
(\nabla_U P)V &= \nabla_U PV - P\nabla_U V, \\
(\nabla_U F)V &= \nabla_U^F V - F\nabla_U V,
\end{align}
\tag{2.8}
\tag{2.9}
\(\nabla, \nabla^\perp\) are symbols used for connections on \(TM\) and \(\gamma M\) respectively while \(h\) and 
\(A_N\) denote the second fundamental forms related by 
\(g(h(U, V), N) = g(A_N U, V),\)
where \(g\) is the Riemannian metric on \(M\) as well as on \(\bar{M}\).

### 3. Semi-Invariant Submanifolds of a Nearly Kenmotsu Manifold

In order to develop the proof of the main theorem we start with the following preparatory results.

**Proposition 3.1.** Let \(M\) be a semi-invariant submanifold of a nearly Kenmotsu manifold \(\bar{M}\) with \(h(X, \phi X) = 0\) for each \(X \in D\). If \(D\) is integrable, then each leaf of the invariant distribution \(D\) is totally geodesic in \(M\) as well as in \(\bar{M}\).

**Proof.** For \(X, Y \in D\), the Gauss equation gives

\[
h(X, \phi Y) + h(\phi X, Y) = (\nabla_X \phi)Y + (\nabla_Y \phi)X + \phi(\nabla_X Y + \nabla_Y X) - (\nabla_X \phi Y + \nabla_Y \phi X).
\]

Using \(h(X, \phi X) = 0\) and nearly Kenmotsu character on \(\bar{M}\), we get

\[
\phi(\nabla_X Y + \nabla_Y X) - (\nabla_X \phi Y + \nabla_Y \phi X) = 0
\]
or,

\[
\phi^2(\nabla_X Y + \nabla_Y X) - \phi(\nabla_X \phi Y + \nabla_Y \phi X) = 0.
\]

In view of (2.1) the above equations becomes

\[
\phi(\nabla_X \phi Y + \nabla_Y \phi X) = -\nabla_X Y - \nabla_Y X + \eta(\nabla_X Y) + \eta(\nabla_Y X) - 2h(X, Y). \quad (3.1)
\]

On the other hand, from equation (2.7) we have \(F\nabla_X X = fh(X, X)\) from which one gets

\[
\nabla_X X \in D \text{ and } h(X, X) \in \phi D^\perp. \quad (3.2)
\]

Replacing \(X\) by \(X + Y\) in the first part of the above observation we get \(\nabla_X Y + \nabla_Y X \in D\). Now taking account of the integrability of \(D\), it follows that

\[
\nabla_X Y \in D. \quad (3.3)
\]

As, \(D\) is integrable, Frobenius theorem guarantees that \(M\) is foliated by leaves of \(D\). Combining this fact with (3.3), we conclude that leaves of \(D\) are totally...
geodesic in $M$. Moreover, making use of (3.3) and (3.1), we get $h(X,Y) = 0$, proving the assertion completely.

In case of totally umbilical semi-invariant submanifold of a nearly Kenmotsu manifold the above proposition yields.

**Corollary 3.1.** Let $M$ be a totally umbilical semi-invariant submanifold of a nearly Kenmotsu manifold $\bar{M}$. Then $M$ is totally geodesic in $\bar{M}$, if $D$ is integrable.

With regard to the anti-invariant distribution, we establish the following.

**Proposition 3.2.** Let $M$ be a totally umbilical semi-invariant submanifold of a nearly Kenmotsu manifold $\bar{M}$ and $\nabla_Z \xi \in \langle \xi \rangle$ for each $Z \in D^\perp$, then anti-invariant distribution $D^\perp$ is integrable and its leaves are totally geodesic in $M$.

**Proof.** Taking $Z \in D^\perp$ and making use of the fact that $\bar{M}$ is nearly Kenmotsu manifold and that $M$ is totally umbilical, equation (2.6) yields that

$$-P \nabla_Z Z = g(H,FZ)Z + g(Z,Z)tH,$$

where $H$ is the mean curvature vector. Obviously the right hand side of the above equation belongs to $D^\perp$ where as the left hand side belongs to $D$, implying that

$$g(H,FZ)Z + g(Z,Z)tH = 0,$$

$$\nabla_Z Z \in D^\perp. \quad (3.5)$$

Equation (3.4) has solution if either

(a) $\dim D^\perp = 1$, or

(b) $H \in \mu$.

If $\dim D^\perp = 1$, then on using (3.5) it follows that $D^\perp$ is integrable and its leaves are totally geodesic in $M$, otherwise we simplify equation (2.6) and obtain the equations $P_X Z = (\nabla_X P)Z$ and $P_Z X = (\nabla_Z P)X$ for $X \in D$. Adding these equations and taking into account that $M$ is nearly Kenmotsu, we get

$$\nabla_Z PX = P(\nabla_Z X + \nabla_X Z),$$

that is $\nabla_Z PX \in D$. The observation $\nabla_Z PX \in D$ under the assumption implies $\nabla_Z W \in D^\perp, \forall Z, W \in D^\perp$. Thus $D^\perp$ is integrable and its leaves are totally geodesic in $M$. This completes the proof.
We are now ready to prove the main theorem.

**Theorem 3.1.** Let $M$ be a totally umbilical semi-invariant submanifold of a nearly Kenmotsu manifold $\bar{M}$. Then at least one of the following is true:

(i) $M$ is anti-invariant.
(ii) $M$ is totally geodesic.
(iii) $\dim D^\perp = 1$ and $D$ is not integrable.

**Proof.** If $D = 0$ then, by definition, $M$ is anti-invariant, which is case (i).
If $D \neq 0$ and integrable, by corollary (3.1) $M$ is totally geodesic which accounts for case (ii). Suppose now $D$ is not integrable and $H \in \mu$, then by virtue of (3.2) $M$ is again totally geodesic. If however, $H \notin \mu$, then equation (3.4) has solution if and only if $\dim D^\perp = 1$ which establishes case (iii). This completes the proof.

It is observed in our preceeding discussion that the integrability of the invariant distribution $D \neq 0$ plays an important role in the geometry of semi-invariant submanifold of nearly Kenmotsu manifold, as the anti-invariant distribution on a totally umbilical semi-invariant submanifold under the condition $\nabla_Z \xi \in \langle \xi \rangle$ $\forall Z \in D^\perp$ is integrable. Thus if we assume that $D$ is integrable, then by Corollary 3.1 its leaves will be totally geodesic in $M$. Thus we have:

**Theorem 3.2.** Let $M$ be a totally umbilical semi-invariant submanifold of a nearly Kenmotsu manifold and $\nabla_Z \xi \in \langle \xi \rangle$ for each $Z \in D^\perp$, then $M$ is locally a Riemannian product of the leaves of the distributions if and only if $D$ is integrable.

**References**


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