GLOBAL ATTRACTIVITY OF AN \textit{n}-SPECIES
NONAUTONOMOUS "PURE-DELAY-TYPE"
COMPETITION SYSTEM

BY

FENGDE CHEN

Abstract. In this paper, we study a non-autonomous \textit{n}-species "pure-delay-type" competitive system, which can be seen as the generalization of the traditional "pure-delay-type" Lotka-Volterra competition system. By constructing a suitable Lyapunov functional, some sufficient conditions which guarantee the global attractivity of the positive solutions of the system are obtained. An example shows the feasibility of our main result.

1. Introduction

The aim of this paper is to obtain sufficient conditions which guarantee the global attractivity of the positive solutions of the following nonautonomous \textit{n}-species "pure-delay-type" competition system

\[
\dot{x}_i(t) = x_i(t) \left[ r_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t) - \sum_{j=1, j \neq i}^{n} b_{ij}(t)x_i(t)x_j(t-\eta_{ij}(t)) \right], \quad (1.1)
\]

where \(x_i(t)(1 \leq i \leq n)\) are the density of the \(i\)-th species. In the theory of mathematical biology, system (1.1) is very important in models which describe the \textit{n}-species population dynamics in a time-fluctuating environment and the effects of time delays. Throughout this paper, it is assumed that:

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\((H_1)\) \(r_i(t), a_{ij}(t), b_{ij}(t)(i \neq j), i, j = 1, 2, \ldots, n\) are continuous and bounded above and below by positive constants on \([0, +\infty)\);

\((H_2)\) \(\tau_{ij}(t), \eta_{ij}(t)(i \neq j)(i, j = 1, \ldots, n)\) are nonnegative bounded continuously differentiable functions on \([0, +\infty)\) and \(\inf_{t \geq 0}(1 - \dot{\tau}_{ij}(t)) > 0, \inf_{t \geq 0}(1 - \dot{\eta}_{ij}(t)) > 0\).

Let \(\tau = \max_{1 \leq i, j \leq n, i \neq j} \{\tau_{ij}(t), \eta_{ij}(t)\}\), then \(\tau < +\infty\). From the viewpoint of mathematical biology, we only consider the initial conditions

\[x_i(s) = \Phi_i(s) \geq 0, \ s \in [-\tau, 0], \ \Phi_i(0) > 0, \ i = 1, 2, \ldots, n, \ (1.2)\]

where \(\Phi = (\Phi_1(\theta), \ldots, \Phi_n(\theta))^T \in C([-\tau, 0], R^n_+)\), the Banach space of continuous functions mapping the interval \([-\tau, 0]\) into \(R^n_+\), where

\[R^n_+ = \{(x_1, x_2, \ldots, x_n)^T : x_i \geq 0, \ i = 1, 2, \ldots, n\}.

It is not difficult to see that solutions of (1.1)-(1.2) are well defined for all \(t \geq 0\) and satisfy

\[x_i(t) > 0, \ \text{for} \ t \geq 0, \ i = 1, 2, \ldots, n.\]

During the last two decades, traditional Lotka-Volterra competition system has been studied extensively (see [1, 5-8, 10, 12, 13, 15, 18, 24, 25, 28, 29]). The model can be expressed as follows:

\[\frac{dx_i(t)}{dt} = x_i(t) \left[r_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t)\right], \ i = 1, 2, \ldots, n. \ (1.3)\]

Many excellent results concerned with the permanence, global asymptotic stability and the existence of positive periodic solutions (almost periodic solutions) in the periodic case (almost periodic case) etc. are obtained.

It is known that if there are time delays in the average growth rates, then (1.3) is modified to a delay differential system of the form

\[\frac{dx_i(t)}{dt} = x_i(t) \left[r_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t - \tau_{ij}(t))\right], \ i = 1, 2, \ldots, n, \ (1.4)\]

where \(r_i(t), a_{ij}(t), i, j = 1, 2, \ldots, n\) are continuous and bounded above and below by positive constants on \([0, +\infty)\), and \(\tau_{ij}(t) (i, j = 1, \ldots, n)\) are nonnegative
bounded continuously differentiable functions on $[0, +\infty)$. Usually, system (1.4) is divided into the following two types: If $\tau_{ii}(t) \equiv 0$, then this type of system (1.4) is called the system of “non-pure-delay-type” and, if $\tau_{ii}(t) \not\equiv 0$, then this type of system (1.4) is called the system of “pure-delay-type”. For the “non-pure-delay-type” system (1.4), since one can use the non-delay term $a_{ii}(t)x_{i}(t)$ to control the delay terms, the analysis of the subjects of the permanence and global attractivity etc. are not so difficult. However, for the pure-delay-type system (1.4), the analysis on the above subjects is so difficult. Recently, R. Xu [27], Z. D. Teng [22-23], X. Z. He [14], F.D. Chen et al. [9], Y. Kuang [17], Z. Cheng et al [11], Y. Xia et al [26] and J. Cao et al [3] had done excellent works on the global attractivity of such kind of system by constructing a suitable Lyapunov functional.

On the other hand, Maynard-Smith [19] incorporated the effect of toxic substances in a two species Lotka-Volterra competitive system by considering that each species produces a substance toxic to the other but only when the other is present, and he proposed the following two-species competition system

\[
\begin{align*}
\dot{x}_1(t) &= x_1(t)[r_1 - a_{11}x_1(t) - a_{12}x_2(t) - b_1x_1(t)x_2(t)], \\
\dot{x}_2(t) &= x_2(t)[r_2 - a_{21}x_1(t) - a_{22}x_2(t) - b_2x_1(t)x_2(t)],
\end{align*}
\]

(1.5)

where $x_1(t), x_2(t)$ are the population density of two competing species and $b_1$ and $b_2$ are, respectively, the rates of toxic inhibition of the first species by the second and vice versa. Chattopadhyay [4] investigated the stability properties of the above system.

Notice that the production of the toxic substance allelopathic to the competing species will not be instantaneous, but delayed by different discrete time lags required for the maturity of both species. Hence, Mukhopadhyay et al [20] proposed the following system

\[
\begin{align*}
\dot{x}_1(t) &= x_1(t)[r_1 - a_{11}x_1(t) - a_{12}x_2(t) - b_1x_1(t)x_2(t - \tau_2)], \\
\dot{x}_2(t) &= x_2(t)[r_2 - a_{21}x_1(t) - a_{22}x_2(t) - b_2x_1(t - \tau_1)x_2(t)],
\end{align*}
\]

(1.6)

where $\tau_i > 0, i = 1, 2$ are the times required for the maturity of the first species and the second species, respectively.
Recently, Song and Chen [21] further considered the nonautonomous case of system (1.6), i.e.,
\[
\dot{x}_1(t) = x_1(t)[r_1(t) - a_{11}(t)x_1(t) - a_{12}(t)x_2(t) - b_1(t)x_1(t)x_2(t - \tau_2(t))],
\]
\[
\dot{x}_2(t) = x_2(t)[r_2(t) - a_{21}(t)x_1(t) - a_{22}(t)x_2(t) - b_2(t)x_1(t - \tau_1(t))x_2(t)].
\]
(1.7)

Under the assumption \(r_i(t), a_{ij}(t), b_i(t), \tau_i(t), i = 1, 2\) are all continuous periodic functions, they obtained the sufficient conditions for the existence of positive periodic solutions of system (1.7). However, they did not investigate the global attractivity of the positive solutions of system (1.7), which is one of the most important topics in the study of mathematical biology. Jin and Ma [16] also did works on this direction.

Obviously systems (1.3)-(1.7) are all special cases of system (1.1). Also, as far as the “pure-delay-type” system is concerned, to the best of the authors’ knowledge, to this day, still no scholars investigate the nonlinear type system. In this paper, by constructing a suitable Lyapunov functional, we will obtain sufficient conditions which guarantee the global attractivity of the positive solutions of system (1.1). In the following we say a positive solution of system (1.1) is globally attractive if it attracts all other positive solutions of the system (1.1).

Throughout this paper, we shall use the following notations:

\begin{itemize}
  \item \(g^l = \min_{t \geq 0} g(t), \quad g^u = \max_{t \geq 0} g(t)\), where \(g\) is a continuous bounded function defined on \([0, +\infty)\).
  \item \(J_i = \{1, \ldots, i - 1, i + 1, \ldots, n\}\).
\end{itemize}

2. Main Results

**Lemma.** Assume that \((H_1)\) and \((H_2)\) hold, and let \(x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T\) denote any solution of system (1.1) with the initial conditions (1.2). Then there exists a \(T > 0\) such that
\[
0 < x_i(t) \leq M_i, \quad \text{for all} \quad t \geq T, \quad i = 1, 2, \ldots, n,
\]
where
\[
M_i = \frac{r_i^u}{a_{ii}^i} \exp \{r_i^u \tau_{ii}^*\}, \quad \text{and} \quad \tau_{ii}^* = \max_{t \in \mathbb{R}} \{\tau_i(t)\}, \quad i = 1, 2, \ldots, n.
\]
**Proof.** Let \( x(t) = (x_1(t), \ldots, x_n(t))^T \) be any solution of system (1.1) with the initial conditions (1.2). Then it follows from the positivity of the solutions and the \( i \)-th equation of system (1.1) that
\[
\dot{x}_i(t) = x_i(t) \left[ r_i(t) - \sum_{j=1}^n a_{ij}(t)x_j(t - \tau_{ij}(t)) - \sum_{j=1, j \neq i}^n b_{ij}(t)x_i(t)x_j(t - \eta_{ij}(t)) \right] \\
\leq x_i(t)[r_i^a - a_{ii}^i x_i(t - \tau_{ii}(t))]. \tag{2.3}
\]
By using (2.3), the rest of the proof is similar to that of Lemma 2.1 of [14] or Theorem 2.1 of [19], and we omit the details here.

Now we state the main result of this section below.

**Theorem.** In addition to (H1)-(H2), assume further that (H3) holds, where (H3) there exist constants \( \lambda_i > 0, i = 1, 2, \ldots, n \) such that for \( i = 1, 2, \ldots, n \)
\[
\liminf_{t \to +\infty} A_i(t) > 0, \quad i = 1, 2, \ldots, n, \tag{2.4}
\]
where
\[
A_i(t) = \lambda_i a_{ii}(t) - \sum_{j \in J_i} \lambda_j \frac{a_{ji}(\zeta_{ji}^{-1}(t))}{1 - \tilde{\tau}_{ji}(\zeta_{ji}^{-1}(t))} - \sum_{j \in J_i} \lambda_j M_j \frac{b_{ji}(\delta_{ji}^{-1}(t))}{1 - \tilde{\eta}_{ji}(\delta_{ji}^{-1}(t))} - \lambda_i \sum_{j \in J_i} b_{ij}(t)M_j \\
- \lambda_i \int_t^\infty a_{ii}(s)ds \left[ r_i(t) + \sum_{j=1}^n a_{ij}(t)M_j + \sum_{j \in J_i} b_{ij}(t)M_jM_j \right] \\
- \sum_{j=1}^n \lambda_j M_j \int_{\zeta_{ji}^{-1}(t)}^{\zeta_{ji}^{-1}(t)} a_{jj}(s)ds \frac{a_{ji}(\zeta_{ji}^{-1}(t))}{1 - \tilde{\tau}_{ji}(\zeta_{ji}^{-1}(t))} \\
- \sum_{j \in J_i} \lambda_j M_j^2 \int_{\delta_{ji}^{-1}(t)}^{\delta_{ji}^{-1}(t)} a_{jj}(s)ds \frac{b_{ji}(\delta_{ji}^{-1}(t))}{1 - \tilde{\eta}_{ji}(\delta_{ji}^{-1}(t))} \\
- \lambda_i \int_t^\infty a_{ii}(s)ds \sum_{j \in J_i} b_{ij}(t)M_iM_j,
\]
where \( \zeta_{ij}^{-1}(t) \) is the inverse function of \( \zeta_{ij}(t) = t - \tau_{ij}(t) \), \( \delta_{ij}^{-1}(t) \) is the inverse function of \( \delta_{ij}(t) = t - \eta_{ij}(t) \). Then for any positive solutions \( x(t) = (x_1(t), \ldots, x_n(t))^T \) and \( y(t) = (y_1(t), \ldots, y_n(t))^T \) of system (1.1), one has
\[
\lim_{t \to +\infty} |x_i(t) - y_i(t)| = 0, \quad i = 1, 2, \ldots, n.
\]
If in system (1.1), $\tau_{ij}(t) \equiv \tau_{ij}, \eta_{ij}(t) \equiv \eta_{ij}$ $(i \neq j)$ $(i, j = 1, 2, \ldots, n)$ are positive constants, then as a direct corollary of Theorem 2.1, we have:

**Corollary.** In addition to $(H_1)-(H_2)$, assume further that $(H'_3)$ holds, where $(H'_3)$ there exist constants $\lambda_i > 0$, $i = 1, 2, \ldots, n$ such that

$$\lim_{t \to +\infty} A'_i(t) > 0, \quad i = 1, 2, \ldots, n,$$  

(2.5)

where

$$A'_i(t) = \lambda_i a_{ii}(t) - \sum_{j \in J_i} \lambda_j a_{ji}(t + \tau_{ji}) - \sum_{j \in J_i} \lambda_j M_j b_{ji}(t + \eta_{ji}) - \lambda_i \sum_{j \in J_i} b_{ij}(t) M_j$$

$$- \lambda_i \int_t^{t+\tau_{ii}} a_{ii}(s) ds \left[ r_i(t) + \sum_{j=1}^n a_{ij}(t) M_j + \sum_{j \in J_i} b_{ij}(t) M_j \right]$$

$$- \lambda_i \sum_{j=1}^n \lambda_j M_j \int_{t+\tau_{ji}}^{t+\tau_{ji}+\tau_{ij}} a_{jj}(s) ds a_{ji}(t + \tau_{ji})$$

$$- \sum_{j \in J_i} \lambda_j M_j \int_{t+\tau_{ji}}^{t+\eta_{ji}+\tau_{ij}} a_{jj}(s) ds b_{ji}(t + \eta_{ji})$$

$$- \lambda_i \int_t^{t+\tau_{ii}} a_{ii}(s) ds \sum_{j \in J_i} b_{ij}(t) M_i M_j.$$

Then for any two positive solutions $x(t) = (x_1(t), \ldots, x_n(t))^T$ and $y(t) = (y_1(t), \ldots, y_n(t))^T$ of system (1.1), one has

$$\lim_{t \to +\infty} |x_i(t) - y_i(t)| = 0.$$

**Proof of Theorem.** Let $x(t) = (x_1(t), \ldots, x_n(t))^T$ and $y(t) = (y_1(t), \ldots, y_n(t))^T$ be any two arbitrary nontrivial solutions of system (1.1)–(1.2). It follows from Lemma 2.1 that there exist positive constants $T$ and $M_i$ $(i = 1, 2, \ldots, n)$ such that

$$0 < x_i(t), \quad y_i(t) \leq M_i, \quad \text{for all } t \geq T, \quad i = 1, 2, \ldots, n.$$  

(2.6)

For $i = 1, 2, \ldots, n$, we let

$$V_{ii}(t) = |\ln x_i(t) - \ln y_i(t)|.$$  

(2.7)
Calculating the upper right derivative of $V_{ii}(t)$ along the solutions of system (1.1), for $t \geq T + \tau$, it follows from (2.6) that

\[
D^+ V_{ii}(t) = \left[ \frac{\dot{x}_i(t)}{x_i(t)} - \frac{\dot{y}_i(t)}{y_i(t)} \right] \text{sgn}(x_i(t) - y_i(t)) \\
= \text{sgn}(x_i(t) - y_i(t)) \left[ - \sum_{j=1}^{n} a_{ij}(t) \left( x_j(t - \tau_{ij}(t)) - y_j(t - \tau_{ij}(t)) \right) \right. \\
- \sum_{j \in J_i} b_{ij}(t) \left( x_i(t)x_j(t - \eta_{ij}(t)) - y_i(t)y_j(t - \eta_{ij}(t)) \right) \\
+ a_{ii}(t) \int_{t-\tau_{ii}(t)}^{t} (\dot{x}_i(u) - \dot{y}_i(u)) \, du \right] \\
= \text{sgn}(x_i(t) - y_i(t)) \left[ - a_{ii}(t)(x_i(t) - y_i(t)) \right. \\
- \sum_{j \in J_i} a_{ij}(t) \left( x_j(t - \tau_{ij}(t)) - y_j(t - \tau_{ij}(t)) \right) \\
- \sum_{j \in J_i} b_{ij}(t) \left( x_i(t)x_j(t - \eta_{ij}(t)) - x_i(t)y_j(t - \eta_{ij}(t)) \right) \\
x_i(t)y_j(t - \eta_{ij}(t)) - y_i(t)y_j(t - \eta_{ij}(t)) \right) \\
+ a_{ii}(t) \int_{t-\tau_{ii}(t)}^{t} \left( x_i(u) \left[ r_i(u) - \sum_{j=1}^{n} a_{ij}(u)x_j(t - \tau_{ij}(u)) \right. \right. \\
- \sum_{j=1,j \neq i}^{n} b_{ij}(u)x_i(u)x_j(u - \eta_{ij}(u)) \right) - y_i(u) \left[ r_i(u) - \sum_{j=1}^{n} a_{ij}(u) \right. \\
\times y_j(u - \tau_{ij}(u)) - \sum_{j=1,j \neq i}^{n} b_{ij}(u)y_i(u)y_j(u - \eta_{ij}(u)) \right) \\
\left. \left. \right] du \right]
\]
\[
\begin{align*}
= \text{sgn}(x_i(t) - y_i(t)) \left[ -a_{ii}(t)(x_i(t) - y_i(t)) \\
- \sum_{j \in J_i} a_{ij}(t) \left( x_j(t - \tau_{ij}(t)) - y_j(t - \tau_{ij}(t)) \right) \\
- \sum_{j \in J_i} b_{ij}(t) \left( x_i(t)(x_j(t - \eta_{ij}(t)) - y_j(t - \eta_{ij}(t))) \right) \\
+ (x_i(t) - y_i(t))y_j(t - \eta_{ij}(t)) \right] + a_{ii}(t) \times \\
\int_{t-\tau_i(t)}^{t} \left( (x_i(u) - y_i(u)) \left[ r_i(u) - \sum_{j=1}^{n} a_{ij}(u)y_j(u - \tau_{ij}(u)) - \\
\sum_{j=1, j \neq i}^{n} b_{ij}(u)y_i(u)y_j(u - \eta_{ij}(u)) \right] \\
+ x_i(u) \left[ -\sum_{j=1}^{n} a_{ij}(u)(x_j(u - \tau_{ij}(u)) - y_j(u - \tau_{ij}(u))) \\
- \sum_{j=1, j \neq i}^{n} b_{ij}(u)(x_i(u)x_j(u - \eta_{ij}(u)) - y_i(u)y_j(u - \eta_{ij}(u))) \right] \right] du \\
\leq -a_{ii}(t)|x_i(t) - y_i(t)| + \sum_{j \in J_i} a_{ij}(t)|x_j(t - \tau_{ij}(t)) - y_j(t - \tau_{ij}(t))| \\
+ \sum_{j \in J_i} b_{ij}(t)M_i|x_j(t - \eta_{ij}(t)) - y_j(t - \eta_{ij}(t))| \\
+ \sum_{j \in J_i} b_{ij}(t)M_j|x_i(t) - y_i(t)| + a_{ii}(t) \int_{t-\tau_i(t)}^{t} \left( |x_i(u) - y_i(u)| \times \\
\left[ r_i(u) + \sum_{j=1}^{n} a_{ij}(u)y_j(u - \tau_{ij}(u)) + \sum_{j \in J_i} b_{ij}(u)y_i(u)y_j(u - \eta_{ij}(u)) \right] \\
+ x_i(u) \sum_{j=1}^{n} a_{ij}(u)|x_j(t - \tau_{ij}(u)) - y_j(t - \tau_{ij}(u))| \\
+ x_i(u) \sum_{j \in J_i} b_{ij}(u)x_i(u)|x_j(u - \eta_{ij}(u)) - y_j(u - \eta_{ij}(u))| \\
+ x_i(u) \sum_{j \in J_i} b_{ij}(u)y_j(u - \eta_{ij}(u))|x_i(u) - y_i(u)| \right] du.
\end{align*}
\]
Define

\[ V_{i2}(t) = \sum_{j \in J_i} \int_{t-T}^{t} \frac{a_{ij}(\zeta^{-1}_{ij}(s))}{1 - \tau_{ij}(\zeta^{-1}_{ij}(s))} |x_j(s) - y_j(s)| ds \]

\[ + \sum_{j \in J_i} M_j \int_{t-\eta_{ij}(t)}^{t} \frac{b_{ij}(\delta^{-1}_{ij}(s))}{1 - \eta_{ij}(\delta^{-1}_{ij}(s))} |x_j(s) - y_j(s)| ds \]

\[ + \int_{t}^{\zeta^{-1}_{ii}(t)} \int_{\zeta_i(s)}^{t} a_{ii}(s) \left( |x_i(u) - y_i(u)| \times \left[ r_i(u) + \sum_{j=1}^{n} a_{ij}(u)y_j(u - \tau_{ij}(u)) + \sum_{j \in J_i} b_{ij}(u)y_i(u)y_j(u - \eta_{ij}(u)) \right] 
\]

\[ + x_i(u) \sum_{j=1}^{n} a_{ij}(u)|x_j(u - \tau_{ij}(u)) - y_j(u - \tau_{ij}(u))| 
\]

\[ + x_i(u) \sum_{j \in J_i} b_{ij}(u)x_i(u)|x_j(u - \eta_{ij}(u)) - y_j(u - \eta_{ij}(u))| 
\]

\[ + x_i(u) \sum_{j \in J_i} b_{ij}(u)y_j(u - \eta_{ij}(u))|x_i(u) - y_i(u)| \right) du ds. \]

Then by using (2.6), for \( t \geq T + \tau \), one has

\[ D^+ V_{i1}(t) + \dot{V}_{i2}(t) \]

\[ \leq -a_{ii}(t)|x_i(t) - y_i(t)| + \sum_{j \in J_i} \frac{a_{ij}(\zeta^{-1}_{ij}(t))}{1 - \tau_{ij}(\zeta^{-1}_{ij}(t))} |x_j(t) - y_j(t)| \]

\[ + \sum_{j \in J_i} M_j \frac{b_{ij}(\delta^{-1}_{ij}(t))}{1 - \eta_{ij}(\delta^{-1}_{ij}(t))} |x_j(t) - y_j(t)| \]

\[ + \sum_{j \in J_i} b_{ij}(t)M_j |x_i(t) - y_i(t)| + \int_{t}^{\zeta^{-1}_{ii}(t)} a_{ii}(s) ds \left( |x_i(t) - y_i(t)| \times \left[ r_i(t) + \sum_{j=1}^{n} a_{ij}(t)y_j(t - \tau_{ij}(t)) + \sum_{j \in J_i} b_{ij}(t)y_i(t)y_j(t - \eta_{ij}(t)) \right] 
\]

\[ + x_i(t) \sum_{j=1}^{n} a_{ij}(t)|x_j(t - \tau_{ij}(t)) - y_j(t - \tau_{ij}(t))| \]
\[+x_i(t) \sum_{j \in J_i} b_{ij}(t)x_i(t)|x_j(t - \eta_j(t)) - y_j(t - \eta_j(t))|\]

\[+x_i(t) \sum_{j \in J_i} b_{ij}(t)y_j(t - \eta_j(t))|x_i(t) - y_i(t)|\]

\[\leq -a_{ii}(t)|x_i(t) - y_i(t)| + \sum_{j \in J_i} \frac{a_{ij}(\zeta_{ij}^{-1}(t))}{1 - \tau_{ij}(\zeta_{ij}^{-1}(t))}|x_j(t) - y_j(t)|\]

\[+ \sum_{j \in J_i} M_i \frac{b_{ij}(\delta_{ij}^{-1}(t))}{1 - \eta_{ij}(\delta_{ij}^{-1}(t))}|x_j(t) - y_j(t)|\]

\[+ \sum_{j \in J_i} b_{ij}(t)M_j|x_i(t) - y_i(t)|\]

\[+ \int_t^{\zeta_{ij}^{-1}(t)} a_{ii}(s)ds \left[ \tau_{ij}(t) + \sum_{j=1}^n a_{ij}(t)M_j + \sum_{j \in J_i} b_{ij}(t)M_j \right]|x_i(t) - y_i(t)|\]

\[+ \int_t^{\zeta_{ij}^{-1}(t)} a_{ii}(s)ds M_i \sum_{j=1}^n a_{ij}(t)|x_j(t - \tau_{ij}(t)) - y_j(t - \eta_{ij}(t))|\]

\[+ \int_t^{\zeta_{ij}^{-1}(t)} a_{ii}(s)ds \sum_{j \in J_i} b_{ij}(t)M_i^2|x_j(t - \eta_{ij}(t)) - y_j(t - \eta_{ij}(t))|\]

\[+ \int_t^{\zeta_{ij}^{-1}(t)} a_{ii}(s)ds \sum_{j \in J_i} b_{ij}(t)M_j|x_i(t) - y_i(t)|\]

We now define

\[V_i(t) = V_{i1}(t) + V_{i2}(t) + V_{i3}(t),\]

in which

\[V_{i3}(t) = M_i \sum_{j=1}^n \int_{t-\tau_{ij}(t)}^t \int_{\zeta_{ij}^{-1}(l)}^{\zeta_{ij}^{-1}(l)} a_{ii}(s) \frac{a_{ij}(\zeta_{ij}^{-1}(l))}{1 - \tau_{ij}(\zeta_{ij}^{-1}(l))} |x_j(l) - y_j(l)| ds \, dl\]

\[+ M_i^2 \sum_{j \in J_i} \int_{t-\eta_{ij}(t)}^{t} \int_{\zeta_{ij}^{-1}(l)}^{\zeta_{ij}^{-1}(l)} a_{ii}(s) \frac{b_{ij}(\delta_{ij}^{-1}(l))}{1 - \eta_{ij}(\delta_{ij}^{-1}(l))} |x_j(l) - y_j(l)| ds \, dl.\]

Then for \( t \geq T + \tau \), one has

\[D^+ V_i(t)\]

\[\leq -a_{ii}(t)|x_i(t) - y_i(t)| + \sum_{j \in J_i} \frac{a_{ij}(\zeta_{ij}^{-1}(t))}{1 - \tau_{ij}(\zeta_{ij}^{-1}(t))}|x_j(t) - y_j(t)|\]
AN n-SPECIES NONAUTONOMOUS “PURE-DELAY-TYPE” COMPETITION SYSTEM

\[ + \sum_{j \in J_i} M_j \frac{b_{ij}(\delta_{ij}^{-1}(t))}{1 - \hat{\sigma}_{ij}(\delta_{ij}^{-1}(t))} |x_j(t) - y_j(t)| \]

\[ + \sum_{j \in J_i} b_{ij}(t) M_j |x_i(t) - y_i(t)| \]

\[ + \int_t^{\zeta_{ij}^{-1}(t)} a_{ii}(s) ds \left[ r_i(t) + \sum_{j=1}^n a_{ij}(t) M_j + \sum_{j \in J_i} b_{ij}(t) M_j M_j \right] |x_i(t) - y_i(t)| \]

\[ + M_i \sum_{j=1}^n \int_{\zeta_{ij}^{-1}(t)}^{\zeta_{ij}^{-1}(\zeta_{ij}^{-1}(t))} a_{ii}(s) ds \frac{a_{ij}(\zeta_{ij}^{-1}(t))}{1 - \bar{\sigma}_{ij}(\zeta_{ij}^{-1}(t))} |x_j(t) - y_j(t)| \]

\[ + M_i^2 \sum_{j \in J_i} \int_{\delta_{ij}^{-1}(t)}^{\zeta_{ij}^{-1}(\delta_{ij}^{-1}(t))} a_{ii}(s) ds \frac{b_{ij}(\delta_{ij}^{-1}(t))}{1 - \bar{\sigma}_{ij}(\delta_{ij}^{-1}(t))} |x_j(t) - y_j(t)| \]

\[ + \int_t^{\zeta_{ij}^{-1}(t)} a_{ii}(s) ds \sum_{j \in J_i} b_{ij}(t) M_i M_j |x_i(t) - y_i(t)|. \]

Now we define a Lyapunov functional as follows:

\[ V(t) = \sum_{i=1}^n \lambda_i V_i(t). \]  

(2.9)

Then it follows from above analysis that for \( t \geq T + \tau \)

\[ D^+ V(t) \leq - \sum_{i=1}^n A_i(t) |x_i(t) - y_i(t)|, \]  

(2.10)

where

\[ A_i(t) = \lambda_i a_{ii}(t) - \sum_{j \in J_i} \lambda_j \frac{a_{ij}(\zeta_{ij}^{-1}(t))}{1 - \bar{\sigma}_{ij}(\zeta_{ij}^{-1}(t))} \]

\[ - \sum_{j \in J_i} \lambda_j M_j \frac{b_{ij}(\delta_{ij}^{-1}(t))}{1 - \bar{\sigma}_{ij}(\delta_{ij}^{-1}(t))} + \lambda_i \sum_{j \in J_i} b_{ij}(t) M_j \]

\[ - \lambda_i \int_t^{\zeta_{ij}^{-1}(t)} a_{ii}(s) ds \left[ r_i(t) + \sum_{j=1}^n a_{ij}(t) M_j - \sum_{j \in J_i} b_{ij}(t) M_j M_j \right] \]

\[ - \sum_{j=1}^n \lambda_j M_j \int_{\zeta_{ij}^{-1}(t)}^{\zeta_{ij}^{-1}(\zeta_{ij}^{-1}(t))} a_{jj}(s) ds \frac{a_{ij}(\zeta_{ij}^{-1}(t))}{1 - \bar{\sigma}_{ij}(\zeta_{ij}^{-1}(t))} \]

\[ - \sum_{j \in J_i} \lambda_j M_j^2 \int_{\delta_{ij}^{-1}(t)}^{\zeta_{ij}^{-1}(\delta_{ij}^{-1}(t))} a_{jj}(s) ds \frac{b_{ij}(\delta_{ij}^{-1}(t))}{1 - \bar{\sigma}_{ij}(\delta_{ij}^{-1}(t))}. \]
By the hypotheses in $(H_3)$, there exist constants $\alpha_i > 0$ ($i = 1, 2, \ldots, n$) and $T^* \geq T + \tau$, such that
\begin{equation}
A_i(t) \geq \alpha_i > 0 \quad \text{for} \quad t \geq T^*, \quad i = 1, 2, \ldots, n.
\end{equation}
From the differential inequality (2.10), one has
\begin{equation}
V(t) - V(T^*) \leq \sum_{i=1}^{n} \int_{T^*}^{t} [A_i(s)|x_i(s) - y_i(s)|] \, ds \quad \text{for} \quad t \geq T^*.
\end{equation}
It follows from (2.11) that
\begin{equation}
V(t) + \alpha \int_{T^*}^{t} \left[ \sum_{i=1}^{n} |x_i(s) - y_i(s)| \right] ds \leq V(T^*) \quad \text{for} \quad t \geq T^*,
\end{equation}
where $\alpha = \min\{\alpha_i, i = 1, 2, \ldots, n\}$. Therefore, $V(t)$ is bounded on $[T^*, +\infty)$ and also
\begin{equation}
\int_{T^*}^{t} \left[ \sum_{i=1}^{n} |x_i(s) - y_i(s)| \right] ds < +\infty.
\end{equation}
By Lemma, $|x_i(t) - y_i(t)|$ ($i = 1, 2, \ldots, n$) are bounded on $[T^*, +\infty)$. On the other hand, it is easy to see that $\dot{x}_i(t)$ and $\dot{y}_i(t)$ ($i = 1, 2, \ldots, n$) are bounded for $t \geq T^*$. Therefore, $|x_i(t) - y_i(t)|$ ($i = 1, 2, \ldots, n$) are uniformly continuous on $[T^*, +\infty)$.
By Barbalat’s Lemma of [2], one can conclude that
\begin{equation}
\lim_{t \to +\infty} \left[ \sum_{i=1}^{n} |x_i(t) - y_i(t)| \right] = 0.
\end{equation}
And so,
\begin{equation}
\lim_{t \to +\infty} |x_i(t) - y_i(t)| = 0, \quad i = 1, 2, \ldots, n.
\end{equation}
The proof of the theorem is complete.

**Example.** We consider the following two species delayed system
\begin{equation}
\begin{cases}
\dot{x}_1(t) = x_1(t) \left[ 2 + \sin t - 8x_1(t - \tau) - (2 + \cos t)x_2(t - \tau_1) - x_1(t)x_2(t - \eta_{12}) \right], \\
\dot{x}_2(t) = x_2(t) \left[ 2 + \cos t - 8x_2(t - \tau) - (2 + \sin t)x_1(t - \tau_2) - x_1(t - \eta_{21})x_2(t) \right].
\end{cases}
\end{equation}
We take $\lambda_1 = \lambda_2 = 1$, then it is easy to examine that

\[ A_1(t) = 8 - (2 + \sin(t + \tau_{21})) - 2M_2 \\
-8\tau[2 + \sin t + 8M_1 + (2 + \cos t)M_2 + M_1M_2] \\
-64M_1\tau - 8M_2\tau(2 + \sin(t + \tau_{21})) - 8M_2^2\tau - 8M_1M_2\tau, \]

\[ A_2(t) = 8 - (2 + \cos(t + \tau_{12})) - 2M_1 \\
-8\tau[2 + \cos t + 8M_2 + (2 + \sin t)M_1 + M_1M_2] \\
-64M_2\tau - 8M_1\tau(2 + \cos(t + \tau_{12})) - 8M_2^2\tau - 8M_1M_2\tau, \]

where $M_1 = M_2 = \frac{3}{8}\exp\{3\tau\}$. Obviously, $A_i(t) > 0$ ($i = 1, 2$) provided that $\tau$ is sufficiently small. Therefore, by Corollary, for any two positive solutions $x(t) = (x_1(t), x_2(t))^T$ and $y(t) = (y_1(t), y_2(t))^T$ of system (2.15), one has

\[ \lim_{t \to +\infty} |x_i(t) - y_i(t)| = 0. \]

3. Conclusion

In this paper, a nonautonomous $n$-species “pure-delay-type” competition system is considered. Attentions are paid to the topic such as boundedness and the global attractivity of the positive solutions of the system. Some interesting results are obtained. Those results have further application on population dynamics.

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College of Mathematics and Computer Science, Fuzhou University, Fuzhou, Fujian 350002, P. R. China.
E-mail: fdchen@fzu.edu.cn; fdchen@263.net