FUZZY PRIME AND FUZZY IRREDUCIBLE IDEALS IN BCK-ALGEBRAS

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Abstract. In this paper, we introduce the concept of fuzzy prime ideals and fuzzy irreducible ideals in BCK-algebras, and investigate its properties.

1. Introduction and preliminaries

The concept of a fuzzy set, which was introduced in [7], is applied in [2] to generalize some of the basic concepts of general topology. Rosenfeld [5] constituted a similar application to the elementary theory of groupoids and groups. Xi [6] applied the concept of fuzzy set to BCK-algebras. In [4], the first author solved the problem of classifying fuzzy ideals by their family of level ideals in BCK(BCI)-algebras.

In the present paper, we introduce the concept of fuzzy prime ideals and fuzzy irreducible ideals in BCK-algebras, and get some results about it.

Recall that a BCK-algebra is a nonempty set $X$ with a binary operation $*$ and a constant $0$ satisfying the axioms:

BCK-1 $((x*y)*(x*z))*(z*y) = 0$,
BCK-2 $(x*(x*y))*y = 0$,
BCK-3 $x*x = 0$,
BCK-4 $0 \ast x = 0$,
BCK-5 $x \ast y = 0$ and $y \ast x = 0$ imply $x = y$,

for all $x, y, z \in X$. We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x \ast y = 0$. A BCK-algebra is said to be commutative if it satisfies the identity $x \ast (y \ast x) = y \ast (y \ast x)$. In this case $y \ast (y \ast x) = x \land y$, the greatest lower bound of $x$ and $y$. A mapping $f : X \to Y$ of BCK-algebras is called a homomorphism if $f(x \ast y) = f(x) \ast f(y)$ for all $x, y \in X$. A nonempty subset $I$ of a BCK-algebra $X$ is called an ideal of $X([3])$ if it satisfies

(I1) $0 \in I$,
(I2) $x \ast y \in I$ and $y \in I$ imply $x \in I$,

for all $x, y \in X$.

We now review some fuzzy logic concepts. We refer the reader to [2], [5] and [6] for complete details. A fuzzy set in a set $X$ is a function $\mu : X \to [0, 1]$. For any fuzzy sets $\mu$ and $\nu$ in a set $X$, we define

$$\mu \leq \nu \iff \mu(x) \leq \nu(x) \quad \text{for all } x \in X,$$

$$(\mu \land \nu)(x) = \min\{\mu(x), \nu(x)\} \quad \text{for all } x \in X.$$

**Definition 1.1.** Let $\mu$ be any fuzzy set in a set $X$. The set

$$\mu_t = \{x \in X : \mu(x) \geq t\}, \quad \text{where} \quad t \in [0, 1],$$

is called a level subset of $\mu$.

**Definition 1.2.** Let $X$ and $X'$ be any two sets, let $\mu$ be any fuzzy set in $X$ and let $f : X \to X'$ be any function. The fuzzy set $\nu$ in $X'$ defined by

$$\nu(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \emptyset, \; y \in X' \\ 0 & \text{otherwise} \end{cases}$$

is called the image of $\mu$ under $f$, denoted by $f(\mu)$.

**Definition 1.3.** Let $X$ and $X'$ be any two sets, let $f : X \to X'$ be any function and let $\nu$ be any fuzzy set in $f(X)$. The fuzzy set $\mu$ in $X$ defined by

$$\mu(x) = \nu(f(x)) \quad \text{for all } x \in X$$

for all $x \in X$. 


is called the preimage of $\nu$ under $f$ and is denoted by $f^{-1}(\nu)$.

**Definition 1.4.** Let $X$ and $X'$ be any two sets, let $f : X \to X'$ be any function. A fuzzy set $\mu$ in $X$ is said to be $f$-invariant if

$$f(x) = f(y) \implies \mu(x) = \mu(y)$$

for all $x, y \in X$.

**Lemma 1.5.** ([2,5]) Let $f$ be a function defined on a set $X$. Then

(a) $\mu \subseteq f^{-1}(f(\mu))$ for any fuzzy set $\mu$ in $X$,
(b) $\mu = f^{-1}(f(\mu))$ provided that $\mu$ is an $f$-invariant fuzzy set in $X$,
(c) $\mu_1 \subseteq \mu_2 \implies f(\mu_1) \subseteq f(\mu_2)$ for any fuzzy sets $\mu_1, \mu_2$ in $X$,
(d) $f(f^{-1}(\nu)) = \nu$ for any fuzzy set $\nu$ in $f(X)$,
(e) $\nu_1 \subseteq \nu_2 \implies f^{-1}(\nu_1) \subseteq f^{-1}(\nu_2)$ for any fuzzy sets $\nu_1, \nu_2$ in $f(X)$.

**Definition 1.6.** A fuzzy set $\mu$ in $X$ has the sup property if, for any subset $T$ of $X$, there exists $x_0 \in T$ such that

$$\mu(x_0) = \sup_{t \in T} \mu(t).$$

**Definition 1.7.** A fuzzy set $\mu$ in a BCK-algebra $X$ is called a fuzzy ideal of $X$ if

(FI1) $\mu(0) \geq \mu(x)$,
(FI2) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$

for all $x, y \in X$.

It is shown in [6] that $\mu$ is a fuzzy ideal if and only if for each $t \in [0,1]$, $\mu_t$ is either empty or an ideal. The ideal $\mu_t$ is called the level ideal of $\mu$.

**Lemma 1.8.** Let $f$ be a homomorphism from a BCK-algebra $X$ onto a BCK-algebra $X'$. Then

(a) if $\mu$ is a fuzzy ideal of $X$ with sup property, then $f(\mu)$ is a fuzzy ideal of $X'$,
(b) if $\mu'$ is a fuzzy ideal of $X'$, then $f^{-1}(\mu')$ is a fuzzy ideal of $X$. 
2. Fuzzy prime ideals

In this section, we define a fuzzy prime ideal in a BCK-algebra, and investigate its properties.

**Definition 2.1.** ([1]) An ideal $I$ in a commutative BCK-algebra $X$ is said to be prime if $x \land y \in I$ implies $x \in I$ or $y \in I$.

**Definition 2.2.** A non-constant fuzzy ideal $\mu$ of a commutative BCK-algebra $X$ is said to be prime if

$$\mu(x \land y) \leq \max\{\mu(x), \mu(y)\}$$

for all $x, y \in X$.

**Lemma 2.3.** ([6]) If $\mu$ is a fuzzy ideal of a BCK-algebra $X$ then the set

$$X_\mu = \{x \in X \mid \mu(x) = \mu(0)\}$$

is an ideal of $X$.

**Theorem 2.4.** Let $\mu$ be a fuzzy prime ideal of a commutative BCK-algebra $X$. Then the set

$$X_\mu = \{x \in X \mid \mu(x) = \mu(0)\}$$

is a prime ideal of $X$.

**Proof.** The fact that $X_\mu$ is an ideal follows from Lemma 2.3. Let $x, y \in X$ be such that $x \land y \in X_\mu$. Then

$$\mu(0) = \mu(x \land y) \leq \max\{\mu(x), \mu(y)\} = \mu(x) \text{ or } \mu(y).$$

It follows from (FI1) that $\mu(x) = \mu(0)$ or $\mu(y) = \mu(0)$. Hence $x \in X_\mu$ or $y \in X_\mu$, which completes the proof.

**Corollary 2.5.** If $\mu$ is a fuzzy prime ideal of a commutative BCK-algebra $X$, then the set

$$P = \{x \in X \mid \mu(x) = 1\}$$
is either empty or a prime ideal of $X$.

**Lemma 2.6.** Let $I$ be an ideal of a BCK-algebra $X$ and let $\alpha < \beta \neq 0$ be elements in $[0,1]$. Then the fuzzy set $\mu : X \to [0,1]$ defined by

$$
\mu(x) = \begin{cases} 
\beta & \text{if } x \in I, \\
\alpha & \text{otherwise}
\end{cases}
$$

is a fuzzy ideal of $X$.

**Proof.** Since $0 \in I$, therefore $\mu(0) = \beta \geq \mu(x)$ for all $x \in X$. Suppose (FI2) does not hold. Then there exist $x, y \in X$ such that $\mu(x) = \alpha$ and $\min\{\mu(x \ast y), \mu(y)\} = \beta$. Thus $\mu(x \ast y) = \beta$ and $\mu(y) = \beta$. Hence $x \ast y \in I$ and $y \in I$. So $x \in I$ since $I$ is an ideal. It leads to a contradiction.

**Theorem 2.7.** Let $P$ be a prime ideal of a commutative BCK-algebra $X$ and let $\alpha \in [0,1]$. If $\mu$ is a fuzzy set in $X$ defined by

$$
\mu(x) = \begin{cases} 
1 & \text{if } x \in P, \\
\alpha & \text{otherwise,}
\end{cases}
$$

then $\mu$ is a fuzzy prime ideal of $X$.

**Proof.** By Lemma 2.6, we know that $\mu$ is a non-constant fuzzy ideal of $X$. Let $x, y \in X$. If $x \wedge y \in P$, then $x \in P$ or $y \in P$. Hence

$$
\mu(x \wedge y) = 1 = \max\{\mu(x), \mu(y)\}.
$$

If $x \wedge y \notin P$ then

$$
\mu(x \wedge y) = \alpha \leq \max\{\mu(x), \mu(y)\}.
$$

Therefore $\mu$ is a fuzzy prime ideal of $X$.

**Corollary 2.8.** If $P$ is a prime ideal of a commutative BCK-algebra $X$, then the characteristic function $\chi_P$ is a fuzzy prime ideal of $X$.

Now we consider the converse of Corollary 2.8.

**Theorem 2.9.** If $P$ is an ideal of a commutative BCK-algebra $X$ such that the characteristic function $\chi_P$ is a fuzzy prime ideal of $X$ then $P$ is a prime ideal of $X$. 
Proof. Let \( x, y \in X \) be such that \( x \wedge y \in P \) and \( x \notin P \). Then
\[
1 = \chi_P(x \wedge y) \leq \max\{\chi_P(x), \chi_P(y)\} = \chi_P(y).
\]
It follows that \( \chi_P(y) = 1 \), so that \( y \in P \). Hence \( P \) is a prime ideal of \( X \).

3. Fuzzy irreducible ideals

In the present section, we introduce the concept of fuzzy irreducible ideals in \( BCK \)-algebras, and give some results.

Definition 3.1. ([1]) A proper ideal \( A \) of a \( BCK \)-algebra \( X \) is said to be irreducible if for any ideals \( I \) and \( J \) of \( X \), \( A = I \cap J \) implies \( I = A \) or \( J = A \).

Definition 3.2. A fuzzy ideal \( \mu \) of a \( BCK \)-algebra \( X \) is said to be fuzzy irreducible if it is not an intersection of two fuzzy ideals of \( X \) properly containing \( \mu \); otherwise \( \mu \) is called fuzzy reducible.

Theorem 3.3. Let \( f \) be a homomorphism from a \( BCK \)-algebra \( X \) onto a \( BCK \)-algebra \( X' \) and let \( \mu \) be a fuzzy irreducible ideal of \( X \) with the sup property. If \( \mu \) is \( f \)-invariant then \( f(\mu) \) is a fuzzy irreducible ideal of \( X' \).

Proof. By Lemma 1.8, \( f(\mu) \) is a fuzzy ideal of \( X' \). Assume that \( f(\mu) \) is fuzzy reducible. Then there exist fuzzy ideals \( \nu \) and \( \sigma \) of \( X' \) such that \( f(\mu) = \nu \cap \sigma, f(\mu) \subseteq \nu, f(\mu) \subseteq \sigma \). As \( \mu \) is \( f \)-invariant, it follows from Lemma 1.5 that
\[
\mu = f^{-1}(\nu \cap \sigma), \quad \mu \subseteq f^{-1}(\nu) \quad \text{and} \quad \mu \subseteq f^{-1}(\sigma).
\]
To show that \( f^{-1}(\nu \cap \sigma) = f^{-1}(\nu) \cap f^{-1}(\sigma) \), let \( x \) be any element of \( X \). Then
\[
(f^{-1}(\nu \cap \sigma))(x) = (\nu \cap \sigma)(f(x))
\]
\[
= \min\{\nu(f(x)), \sigma(f(x))\}
\]
\[
= \min\{(f^{-1}(\nu))(x), (f^{-1}(\sigma))(x)\}
\]
\[
=(f^{-1}(\nu) \cap f^{-1}(\sigma))(x),
\]
which implies that  
\[ f^{-1}(\nu \cap \sigma) = f^{-1}(\nu) \cap f^{-1}(\sigma). \]

Hence \( \mu = f^{-1}(\nu) \cap f^{-1}(\sigma), \mu \subset f^{-1}(\nu), \mu \subset f^{-1}(\sigma) \). This contradicts the fact that \( \mu \) is fuzzy irreducible. This completes the proof.

**Theorem 3.4.** Let \( f \) be as in Theorem 3.3 and let \( \mu' \) be any fuzzy irreducible ideal of \( X' \). If every fuzzy ideal of \( X \) is \( f \)-invariant, then \( f^{-1}(\mu') \) is a fuzzy irreducible ideal of \( X \).

**Proof.** By Lemma 1.8, \( f^{-1}(\mu') \) is a fuzzy ideal of \( X \). Suppose that \( f^{-1}(\mu') \) is fuzzy reducible. Then \( f^{-1}(\mu') = \sigma \cap \theta, f^{-1}(\mu') \subset \sigma \) and \( f^{-1}(\mu') \subset \theta \) for some fuzzy ideals \( \sigma \) and \( \theta \) of \( X \). It is evident from Lemma 1.5 that \( \mu' = f(\sigma \cap \theta), \mu' \subset f(\sigma) \) and \( \mu' \subset f(\theta) \). Now we show that \( f(\sigma \cap \theta) = f(\sigma) \cap f(\theta) \). Since \( \sigma \cap \theta \subset \sigma \) and \( \sigma \cap \theta \subset \theta \), it follows from Lemma 1.5(c) that \( f(\sigma \cap \theta) \subset f(\sigma) \cap f(\theta) \). To establish the reverse inclusion, let \( y \in X', \alpha = (f(\sigma) \cap f(\theta))(y) \) and \( \epsilon > 0 \) be any real number. Then

\[
\alpha - \epsilon < \min\{ (f(\sigma))(y), (f(\theta))(y) \} \\
= \min\{ \sup_{x \in f^{-1}(y)} \sigma(x), (f(\theta))(y) \},
\]

which implies that \( \alpha - \epsilon < \sigma(z) \) for some \( z \in f^{-1}(y) \) and \( \alpha - \epsilon < (f(\theta))(y) \). This means \( \alpha - \epsilon < \sigma(z) \) and

\[
\alpha - \epsilon < (f(\theta))(f(z)) \\
= (f^{-1}(f(\theta)))(z) \\
= \theta(z) \quad \text{since } \theta \text{ is } f\text{-invariant.}
\]

Hence

\[
\alpha - \epsilon < \min\{ \sigma(z), \theta(z) \} = (\sigma \cap \theta)(z).
\]

From \( z \in f^{-1}(y) \) it follows that

\[
\sigma - \epsilon < \sup_{x \in f^{-1}(y)} (\sigma \cap \theta)(x) = (f(\sigma \cap \theta))(y).
\]
As $\epsilon > 0$ was arbitrary, therefore

$$\alpha = (f(\sigma) \cap f(\theta))(y) \leq (f(\sigma \cap \theta))(y),$$

so that

$$f(\sigma) \cap f(\theta) \subseteq f(\sigma \cap \theta).$$

Hence $\mu' = f(\sigma) \cap f(\theta), \mu' \subseteq f(\sigma)$ and $\mu' \subseteq f(\theta)$, whence $\mu'$ is fuzzy reducible, a contradiction. The proof is complete.

The following theorem is an immediate consequence of Theorems 3.3 and 3.4.

**Theorem 3.5.** Let $f$ be a homomorphism form a BCK-algebra $X$ onto a BCK-algebra $X'$, and let every fuzzy ideal of $X$ be $f$-invariant. Then the mapping $\theta \mapsto f(\theta)$ defines an one-to-one correspondence between the set of all fuzzy irreducible ideals of $X$ with the sup property and the set of all fuzzy irreducible ideals of $X'$.

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**References**


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