Abstract. After the works of Bernal [2] on the idea of relative order of entire functions, Lahiri and Banerjee [6] introduced the definition of relative order of a meromorphic function. In this note we consider relative \((p, q)\) order of meromorphic functions where \(p\) and \(q\) are positive integers with \(p > q\) when the related functions are entire. Proving several basic theorems, we show that relative \((p, q)\) order remains unchanged for derivatives. Further we observe that the finiteness of relative \((p, q)\) order is closely connected with the convergence of a certain integral.

1. Introduction and Definitions

Let \(f\) and \(g\) be entire functions and \(F(r) = \max\{|f(z)| : |z| = r\}\) and \(G(r) = \max\{|g(z)| : |z| = r\}\). If \(f\) is non-constant, then \(F(r)\) is strictly increasing and continuous function of \(r\) and its inverse

\[ F^{-1} : ([f(0)], \infty) \to (0, \infty) \]

exists and \(\lim_{R \to \infty} F^{-1}(R) = \infty\).

In the paper [2] Bernal introduced the definition of relative order of \(f\) with respect to \(g\) as follows.

Definition 1. Let \(f\) and \(g\) be non-constant entire functions. Then the relative order of \(f\) with respect to \(g\), denoted by \(\rho_g(f)\), is defined by

\[ \rho_g(f) = \inf\{\mu > 0 : F(r) < G(r^\mu) \text{ for all } r > r_0(\mu) > 0\}. \]
Definition 1 coincides with the classical definition ([11], p.248) of $f$ if $g(z) = \exp z$.

If $f$ is meromorphic, in a recent paper [6] Lahiri and Banerjee defined the relative order of $f$ with respect to $g$ as follows.

**Definition 2.** Let $f$ be meromorphic and $g$ be entire. The relative order of $f$ with respect to $g$ is defined by

$$
\rho_g(f) = \inf \{ \mu > 0 : T_f(r) < T_g(r^\mu) \text{ for all large } r \}.
$$

Here $T_f(r) = T(r, f)$ etc. denote the Nevanlinna characteristic function ([4]).

It is known ([6]) that if $g(z) = \exp z$ then Definition 2 coincides with the classical definition of order of a meromorphic function $f$.

Following Sato [9], we write $\log^0 x = x$, $\exp^0 x = x$ and for a positive integer $m \geq 1$, $\log^m x = \log(\log^{m-1} x)$, $\exp^m x = \exp(\exp^{m-1} x)$. Note that $\log^m x$ is defined for all $x > 0$ large enough, namely, for all $x > \exp^{m-1} 0$.

In the recent paper [7] Lahiri and Banerjee introduced the definition of relative order $(p, q)$ of entire funtions which is as follows:

Let $p$ and $q$ be positive integers with $p > q$. The relative order $(p, q)$ of $f$ with respect to $g$ is defined by

$$
\rho_{p,q}(f, g) = \inf \{ \mu > 0 : F(r) < G(\exp^{p-1}(\mu \log^q r)), \text{ for all } r > r_0(\mu) > 0 \}.
$$

In the study of relative order, it therefore seems reasonable to define suitably relative $(p, q)$ order of meromorphic functions and to investigate its basic properties. This is our goal in this paper.

We now introduce relative $(p, q)$ order of meromorphic $f$ with respect to entire $g$ as follows.

**Definition 3.** Let $f$ and $g$ be non-constant functions of which $f$ is meromorphic and $g$ is entire and let $p$, $q$ be positive integers with $p > q$. Then the relative $(p, q)$ order of $f$ with respect to $g$, denoted by $\rho_{p,q}(f, g)$, is defined by

$$
\rho_{p,q}(f, g) = \inf \{ \mu > 0 : T_f(r) < T_g(\exp^{p-1}(\mu \log^q r)), \text{ for all } r > r_0(\mu) > 0 \}.
$$

If $p = 2$, $q = 1$, then $\rho_{p,q}(f, g) = \rho_g(f)$. If $g(z) = \exp z$, then $\rho_{p,q}(f, g)$ coincides with $(p, q)$th order of $f$ [1, 10].
This paper is organized as follows. In Section 2 we recall a concept as well as a number of preparatory results, and an alternative expression for the relative \((p, q)\) order is provided. Sections 3 and 4 are devoted to study the behavior of the relative \((p, q)\) order in front of the usual operations of sum and product; and to establish its invariance under derivation, whenever suitable conditions are satisfied. Finally, in Section 5 it is shown how the finiteness of \(\rho_{p,q}(f, g)\) is closely connected with the convergence of a certain integral related to the growth of the two functions \(f, g\). Throughout the present paper we shall assume \(f, g\) etc. are non-constant.

2. Property \((A)\) and Preliminary Results

First, the following definition is recalled.

**Definition 4.** ([2]) An entire function \(g\) is said to have the property \((A)\) if for any \(\sigma > 1\) and for all large \(r\)

\[|G(r)|^2 \leq G(r^\sigma)\]

holds.

Examples of entire functions are known ([2]) which have the property \((A)\) as well as which do not have the property \((A)\).

The following five lemmas will be needed in the subsequent sections.

**Lemma 1.** ([2]) Let \(g\) be an entire function which has the property \((A)\). Then for any positive integer \(n\) and for all \(\sigma > 1\) and for all large \(r\), the inequality

\[|G(r)|^n \leq G(r^\sigma)\]

holds.

**Lemma 2.** ([4, p.18]) If \(g\) is entire then

\[T_g(r) \leq \log G(r) \leq 3T_g(2r)\] for all large \(r\).

**Lemma 3.** ([8]) If \(f(z)\) is a transcendental meromorphic function then

\[T_f(r) \leq 2T_f(2r) + O\{T_f(2r)\}\] for all large values of \(r\).
Lemma 4. ([3], [12]) Let $f$ be a meromorphic function. Then for all large values of $r$

$$T_f(r) < C(T_f(2r) + \log r),$$

where $C$ is a constant which is only dependent on $f(0)$.

Lemma 5. ([2]) If $g$ is entire transcendental and $s > 1$ then for any positive integer $n$

$$\lim_{r \to \infty} \frac{G(r^s)}{r^n G(r)} = \infty.$$

To finish, we give another expression for the relative $(p, q)$ order, which sometimes is easier to handle.

Theorem 1. Let $f$ be meromorphic and $g$ be entire. Then

$$\rho_{p,q}(f, g) = \limsup_{r \to \infty} \frac{\log^{[p-1]} T_g^{-1} T_f(r)}{\log^{[q]} r}.$$

Proof. Suppose that $\rho_{p,q}(f, g)$ is finite. If $\epsilon > 0$, $\exists r_0(\epsilon) > 0$ such that

$$T_f(r) < T_g\left(\exp^{[p-1]} \left(\rho_{p,q}(f, g) + \epsilon \log^{[q]} r\right)\right)$$

for $r \geq r_0$.

$$\therefore \frac{\log^{[p-1]} T_g^{-1} T_f(r)}{\log^{[q]} r} < \rho_{p,q}(f, g) + \epsilon \quad \text{for } r \geq r_1 \geq r_0. \quad (1)$$

Also there exists a sequence $r_n \to \infty$ such that

$$\frac{\log^{[p-1]} T_g^{-1} T_f(r_n)}{\log^{[q]} r_n} > \rho_{p,q}(f, g) - \epsilon \quad \text{for all large } n. \quad (2)$$

(1) and (2) give

$$\rho_{p,q}(f, g) = \limsup_{r \to \infty} \frac{\log^{[p-1]} T_g^{-1} T_f(r)}{\log^{[q]} r}.$$ 

If $\rho_{p,q}(f, g) = \infty$, this easily follows from the definition.

3. Sum and Product Theorems

Here we study the relative $(p, q)$ order of the sum and the product of two meromorphic functions with respect to the same entire function.
Theorem 2. Let $f_1$ and $f_2$ be meromorphic functions having relative $(p,q)$ orders $\rho_{p,q}(f_1,g)$ and $\rho_{p,q}(f_2,g)$ respectively, where $g$ has the property (A). Then (i) $\rho_{p,q}(f_1 \pm f_2, g) \leq \max\{\rho_{p,q}(f_1, g), \rho_{p,q}(f_2, g)\}$ and (ii) $\rho_{p,q}(f_1 f_2, g) \leq \max\{\rho_{p,q}(f_1, g), \rho_{p,q}(f_2, g)\}$. The equality holds in (ii) if $\rho_{p,q}(f_1, g) \neq \rho_{p,q}(f_2, g)$.

For the quotient we have also $\rho_{p,q}(\frac{f_1}{f_2}, g) \leq \max\{\rho_{p,q}(f_1, g), \rho_{p,q}(f_2, g)\}$.

Proof. We may suppose that $\rho_{p,q}(f_1, g)$ and $\rho_{p,q}(f_2, g)$ both are finite, because if one of $\rho_{p,q}(f_1, g)$, $\rho_{p,q}(f_2, g)$ or both are infinite, the inequalities are evident. Let

$$\rho_1 = \rho_{p,q}(f_1, g), \ \rho_2 = \rho_{p,q}(f_2, g), \ \text{and} \ \rho_1 \leq \rho_2.$$ 

Then for arbitrary $\epsilon > 0$ and for all large $r$ we have

$$T_{f_1}(r) < T_g(\exp^{[p-1]}(\rho_1 + \frac{\epsilon}{4}) \log^{[q]} r) \leq T_g(\exp^{[p-1]}(\rho_2 + \frac{\epsilon}{4}) \log^{[q]} r)$$

and

$$T_{f_2}(r) < T_g(\exp^{[p-1]}(\rho_2 + \frac{\epsilon}{4}) \log^{[q]} r).$$

So for all large $r$,

$$T_{f_1 \pm f_2}(r) \leq T_{f_1}(r) + T_{f_2}(r) + O(1)$$

$$\leq T_g(\exp^{[p-1]}(\rho_2 + \frac{\epsilon}{4}) \log^{[q]} r) + T_g(\exp^{[p-1]}(\rho_2 + \frac{\epsilon}{4}) \log^{[q]} r) + O(1)$$

$$\leq 3 \log G(\exp^{[p-1]}(\rho_2 + \frac{\epsilon}{4}) \log^{[q]} r), \ \text{using Lemma 2}$$

$$= \frac{1}{3} \log G(\exp^{[p-1]}(\rho_2 + \frac{\epsilon}{4}) \log^{[q]} r)^9$$

$$\leq \frac{1}{3} \log G(\exp^{[p-1]}(\rho_2 + \frac{\epsilon}{4}) \log^{[q]} r)^\sigma, \ \text{for every} \ \sigma > 1, \ \text{by Lemma 1},$$

$$= \frac{1}{3} \log G\left(\exp^{[p-2]}(\log^{[q-1]} r)^{\rho_2 + \frac{\epsilon}{2}}\right)^\sigma.$$ \hspace{1cm} (3)

Consider the expression $\frac{\exp^{[p-3]}(\log^{[q-1]} r)^{\rho_2 + \frac{\epsilon}{2}}}{\exp^{[p-3]}(\log^{[q-1]} r)^{\rho_2 + \frac{\epsilon}{4}}}$, $p \geq 3$ which is greater than 1 if $r > \exp^{[q]} 0$ and tends to $\infty$ as $r \to \infty$. So for all large $r$, say $r \geq r_1 > r_0$ the above expression is greater than $\frac{\exp^{[p-3]}(\log^{[q-1]} r_0)^{\rho_2 + \frac{\epsilon}{2}}}{\exp^{[p-3]}(\log^{[q-1]} r_0)^{\rho_2 + \frac{\epsilon}{4}}}$ which itself greater than 1.

Suppose $\sigma = \frac{\exp^{[p-3]}(\log^{[q-1]} r_0)^{\rho_2 + \frac{\epsilon}{2}}}{\exp^{[p-3]}(\log^{[q-1]} r_0)^{\rho_2 + \frac{\epsilon}{4}}}$, then $\sigma > 1$ and from (3), we have

$$T_{f_1 \pm f_2}(r) \leq \frac{1}{3} \log G(\exp^{[p-1]}(\rho_2 + \frac{\epsilon}{4}) \log^{[q]} r)^\sigma$$

$$\leq \frac{1}{3} \log G(\exp^{[p-2]}(\log^{[q-1]} r)^{\rho_2 + \frac{\epsilon}{2}}).$$
for the above value of $\sigma$ and for all $r, r \geq r_1 > r_0$

\[
\leq T_g \left( 2 \exp^{[p-1]}(\rho_2 + \frac{\epsilon}{2}) \log^{|q|} r \right) \text{ by Lemma 2}
\]

\[
\leq T_g \left( \exp^{[p-1]}(\rho_2 + \epsilon) \log^{|q|} 2r \right).
\]

Since $\epsilon > 0$ is arbitrary, we obtain

\[
\rho_{p,q}(f_1 \pm f_2, g) \leq \rho_2 = \max\{\rho_{p,q}(f_1, g), \rho_{p,q}(f_2, g)\}
\]

which proves (i).

For (ii), since $T_{f_1 f_2}(r) \leq T_{f_1}(r) + T_{f_2}(r)$, we obtain similarly as above

\[
\rho_{p,q}(f_1 f_2, g) \leq \max\{\rho_{p,q}(f_1, g), \rho_{p,q}(f_2, g)\}.
\]

Now we prove the inequality (ii) for the quotient. Let $f = f_1 f_2$ and suppose $\rho_1 \neq \rho_2$. We show that in (4), the equality will hold. For this, let $f = f_1 f_2$ and $\rho_{p,q}(f_1, g) < \rho_{p,q}(f_2, g)$. Then $\rho_{p,q}(f, g) \leq \rho_{p,q}(f_2, g)$. Again since $f_2 = \frac{f_1}{f_2}$ and $T_{f_1}(r) = T_{f_2}(r) + O(1)$, applying (4) we have

\[
\rho_{p,q}(f_2, g) \leq \max\{\rho_{p,q}(f, g), \rho_{p,q}(f_1, g)\}. \quad \text{Since } \rho_{p,q}(f_1, g) < \rho_{p,q}(f_2, g), \text{ we have}
\]

\[
\rho_{p,q}(f_2, g) \leq \rho_{p,q}(f, g) \quad \text{and so } \rho_{p,q}(f, g) = \rho_{p,q}(f_2, g) = \max\{\rho_{p,q}(f_1, g), \rho_{p,q}(f_2, g)\}.
\]

Now we prove the inequality (ii) for the quotient. Let $f = f_1 f_2$ and suppose $\rho_1 \leq \rho_2$. Then $f_1 = f f_2$. If possible let $\rho_{p,q}(f, g) > \rho_2$. Applying case (ii) we obtain $\rho_1 = \rho_{p,q}(f, g)$ and so $\rho_1 > \rho_2$, which contradicts our hypothesis. So

\[
\rho_{p,q}(f_2, g) \leq \rho_{p,q}(f, g) = \max\{\rho_{p,q}(f_1, g), \rho_{p,q}(f_2, g)\}.
\]

Now suppose $\rho_1 < \rho_2$. If possible, let $\rho_{p,q}(f, g) < \rho_2$. Then $\rho_1 = \max\{\rho_{p,q}(f, g), \rho_2\} = \rho_2$. Which is also a contradiction.

\[
\therefore \rho_{p,q}(f_1 f_2, g) \leq \rho_{p,q}(f_2, g) = \max\{\rho_{p,q}(f_1, g), \rho_{p,q}(f_2, g)\}.
\]

This proves the theorem.

**Note 1.** When $p = 2$, the proof of the theorem follows from Lahiri and Banerjee [6].

Finally, we want to point out an alternative definition of $\rho_{p,q}(f, g)$ in the case of an entire function $g$ satisfying $\langle A \rangle$. The proof is omitted and left to the interested reader.
Theorem 3. Let \( f \) be a meromorphic function having relative \((p, q)\) order \( \rho_{p,q}(f, g) \), where \( g \) is entire. Then for an arbitrary \( \epsilon > 0 \), \( T_f(r) = O[\log G(\exp^{[p-1]}(\rho_{p,q}(f, g) + \epsilon) \log^q r)] \) holds for all large \( r \). Conversely, if for a meromorphic \( f \) and entire \( g \) having the property \((A)\), \( T_f(r) = O[\log G(\exp^{[p-1]}(k + \epsilon) \log^q r)] \) holds for all large \( r \) and \( T_f(r) = O[\log G(\exp^{[p-1]}(k - \epsilon) \log^q r)] \) does not hold for all large values of \( r \) then \( k = \rho_{p,q}(f, g) \).

4. Relative \((p, q)\) Order of the Derivative

Regarding the relative \((p, q)\) order of \( f \) and its derivative \( f' \) with respect to \( g \) and \( g' \) we prove the following theorems.

Theorem 4. Let \( f \) be a transcendental meromorphic function and \( g \) be an entire function with the property \((A)\). Then \( \rho_{p,q}(f, g) = \rho_{p,q}(f', g) \).

Proof. From Lemmas 3 and 4, we have for all large \( r \)

\[
T_{f'}(r) < [K]T_f(2r), \text{ where } K > 1
\]

and \( T_f(r) < [K']T_{f'}(2r), \text{ where } K' > 1. \)

For arbitrary \( \epsilon > 0 \),

\[
T_f(2r) < T_g\left( \exp^{[p-1]}\left( \rho_{p,q}(f, g) + \frac{\epsilon}{4} \right) \log^q 2r \right) \text{ for all large } r.
\]

Using Lemmas 1 and 2, for all large \( r \) we have from (5)

\[
T_{f'}(r) < \frac{1}{3} \log \left[ G\left( \exp^{[p-1]}\left( \rho_{p,q}(f, g) + \frac{\epsilon}{4} \right) \log^q 2r \right) \right]^{3[K]}
\]

\[
\leq \frac{1}{3} \log G\left( \exp^{[p-1]}\left( \rho_{p,q}(f, g) + \frac{\epsilon}{4} \right) \log^q 2r \right) \sigma \text{ for every } \sigma > 1.
\]

Now proceeding similarly like after (3) we arrive at

\[
T_{f'}(r) < T_g\left( \exp^{[p-1]}\left( \rho_{p,q}(f, g) + \epsilon \right) \log^q 4r \right).
\]

Since \( \epsilon > 0 \) is arbitrary, we get by Theorem 1 that

\[
\rho_{p,q}(f', g) = \limsup_{r \to \infty} \frac{\log^{[p-1]} T_{f'}(r)}{\log^q r} \leq \rho_{p,q}(f, g).
\]
Similarly using (6) we obtain
\[ \rho_{p,q}(f, g) \leq \rho_{p,q}(f', g) \] and this proves the theorem.

**Note 2.** Theorem 4 is still true even when \( f \) is a rational meromorphic function. The following lemma gives the assertion.

**Lemma 6.** Let \( f \) be a rational meromorphic function, then for all large values of \( r \), \( T_f(r) < T_f(r) + K \log r + O(1) \) where \( K \) is a constant depending on \( f \) only.

**Proof.** In this case we write \( f(z) \) as
\[ f(z) = \frac{\phi(z)}{\psi(z)} \]
where \( \phi(z) \) and \( \psi(z) \) are polynomials. Taking logarithmic differentiation we can obtain
\[ f' = f \left( \frac{\phi'}{\phi} - \frac{\psi'}{\psi} \right) \]
and so
\[ T_f(r) \leq T_f(r) + T_{\phi}(r) + T_{\psi}(r) + \log 2 \]
\[ = T_f(r) + (\lambda + \mu) \log r + O(1) + \log 2 \]
where \( \lambda \) and \( \mu \) are degrees of the polynomials \( \phi \) and \( \psi \) respectively
\[ = T_f(r) + K \log r + O(1). \]
This proves the lemma.

**Theorem 5.** Let \( f \) be meromorphic and \( g \) be entire transcendental having the property \( (A) \), then \( \rho_{p,q}(f, g) = \rho_{p,q}(f, g') \).

The following lemma is required.

**Lemma 7.** Let \( g \) be entire transcendental and \( \bar{G}(r) = \max\{|g'(z)| : |z| = r\} \). Then
\[ G(r^\lambda) < \bar{G}(r) < G(2r), \ r > 1 \text{ and } \lambda \in (0, 1). \]
Proof. We may write \( g(z) = \int_0^z g'(t) dt \), where the line of integration is the segment from \( z = 0 \) to \( z = re^{i\theta_0}, r > 0 \). Let \( z_1 = re^{i\theta_1} \) be such that \( |g(z_1)| = \max\{|g(z)| : |z| = r\} \).

Then \( G(r) = |g(z_1)| = \left| \int_0^{z_1} g'(t) dt \right| \leq r \max\{|g'(z)| : |z| = r\} = rG(r). \) (7)

Let \( C \) denote the circle \( |t - z_0| = r \), where \( z_0, |z_0| = r \) is defined so that \( |g'(z_0)| = \max\{|g'(z)| : |z| = r\} \). So

\[
G(r) = \max\{|g'(z)| : |z| = r\} = |g'(z_0)| = \left| \frac{1}{2\pi i} \oint_C \frac{g(t)}{(t - z_0)^2} dt \right| \leq \frac{G(2r)}{r}. \] (8)

From (7) and (8)

\[
\frac{G(r)}{r} \leq G(r) \leq \frac{G(2r)}{r}, \quad r > 0. \] (9)

Let \( \lambda \in (0, 1) \) and \( s = \frac{1}{\lambda} \). Let \( n \) be a positive integer such that \( n\lambda \geq 1 \). Since \( g \) is transcendental, from Lemma 5, \( G(r^s) > r^n G(r) \) for all large \( r \).

If we replace \( r \) by \( r^\lambda \), then from above

\[
G(r^\lambda) > r^n \lambda G(r^\lambda) \geq rG(r^\lambda) \text{ i.e., } G(r) > rG(r^\lambda). \]

From (9), \( G(r^\lambda) < \frac{G(r)}{r} \leq G(r) \leq \frac{G(2r)}{r} < G(2r), \quad r > 1 \)

i.e., \( G(r^\lambda) < G(r) < G(2r), \quad r > 1 \).

This proves the lemma.

Proof of Theorem 5. We can assume that \( \rho_{p,q}(f, g) \) and \( \rho_{p,q}(f, g') \) are
both finite. By the definition of $\rho_{p,q}(f, g)$, for all large $r$ and for arbitrary $\epsilon > 0$

\[
T_f(r) < T_g\left( \exp^{[p-1]}(\mu_1 + \epsilon) \log^{[q]} r \right) \quad \text{where } \rho_{p,q}(f, g) < \mu_1
\]

\[
\leq \frac{1}{3} \log \left[ G\left( \exp^{[p-1]}(\mu_1 + \epsilon) \log^{[q]} r \right) \right] \quad \text{using Lemma 2}
\]

\[
< \frac{1}{3} \log G\left( \exp^{[p-1]}(\mu_1 + \epsilon \log^{[q]} r) \right)^{\sigma} \quad \text{for any } \sigma > 1, \text{ by Lemma 1}
\]

\[
\leq \frac{1}{3} \log G\left( \exp^{[p-1]}(\mu_1 + 2\epsilon \log^{[q]} 2r) \right)
\]

\[
< \frac{1}{3} \log G\left( \exp^{[p-1]}(\mu_1 + 2\epsilon \log^{[q]} 2r) \right)^{\lambda} \quad \text{where } \lambda \in (0, 1), \text{ by Lemma 7}
\]

\[
< \frac{1}{3} \log G\left( \frac{1}{2} \exp^{[p-1]}(\mu_1 + 4\epsilon \log^{[q]} 4r) \right) \quad \text{where } \mu_1 < \mu
\]

\[
\leq T_{g'}\left( \exp^{[p-1]}(\mu_1 + 4\epsilon \log^{[q]} 4r) \right) \quad \text{using Lemma 2, since } g' \text{ is entire}
\]

\[
\therefore \limsup_{r \to \infty} \frac{\log^{[p-1]} T_f^{-1}(r)}{\log^{[q]} r} \leq \frac{\mu + 4\epsilon}{\lambda}.
\]

Since $\lambda \in (0, 1)$ and $\epsilon > 0$ is arbitrary, we obtain $\rho_{p,q}(f, g') \leq \mu$.

Now $\mu > \mu_1 > \rho_{p,q}(f, g)$ is arbitrary, so we have finally

\[
\rho_{p,q}(f, g') \leq \rho_{p,q}(f, g). \quad (10)
\]

To obtain the converse inequality, from the definition of $\rho_{p,q}(f, g')$ we have for all large $r$

\[
T_f(r) < T_{g'}\left( \exp^{[p-1]}(\lambda_1 + \epsilon) \log^{[q]} r \right) \quad \text{where } \rho_{p,q}(f, g') < \lambda_1
\]

\[
\leq \log G\left( \exp^{[p-1]}(\lambda_1 + \epsilon \log^{[q]} r) \right) \quad \text{using Lemma 2, since } g' \text{ is entire}
\]

\[
< \log G\left( 2 \exp^{[p-1]}(\lambda_1 + \epsilon \log^{[q]} r) \right) \quad \text{by Lemma 7}
\]

\[
< \log G\left( \exp^{[p-1]}(\lambda_1 + 2\epsilon \log^{[q]} 2r) \right)
\]

\[
< \log G\left( \exp^{[p-1]}(\lambda_2 + 2\epsilon \log^{[q]} 2r) \right) \quad \text{for } \lambda_2 > \lambda_1
\]

\[
= \frac{1}{3} \log \left[ G\left( \exp^{[p-1]}(\lambda_2 + 2\epsilon \log^{[q]} 2r) \right) \right]^3
\]
\[
\leq \frac{1}{3} \log G \left( \exp^{[p-1]}(\lambda_2 + 2\epsilon \log [q] 2r) \right)^{\sigma} \text{ for any } \sigma > 1, \text{ by Lemma 1}
\]
\[
\leq \frac{1}{3} \log G \left( \exp^{[p-1]}(\lambda_2 + 4\epsilon \log [q] 4r) \right)
\]
\[
< \frac{1}{3} \log G \left( \frac{1}{2} \exp^{[p-1]}(\lambda + 4\epsilon \log [q] 4r) \right) \text{ where } \lambda_2 < \lambda
\]
\[
\leq T_g \left( \exp^{[p-1]}(\lambda + 4\epsilon \log [q] 4r) \right) \text{ by Lemma 2}
\]
\[
\therefore \limsup_{r \to \infty} \frac{\log^{[p-1]} T_g^{-1} T_f(r)}{\log[q] r} \leq \lambda + 4\epsilon.
\]
Since \( \epsilon > 0 \) is arbitrary, we obtain \( \rho_{p,q}(f, g) \leq \lambda \).

Now since \( \lambda > \rho_{p,q}(f, g) \) is arbitrary, we obtain
\[
\rho_{p,q}(f, g) \leq \rho_{p,q}(f, g').
\]
Combining (10) and (11), we obtain
\[
\rho_{p,q}(f, g) = \rho_{p,q}(f, g')
\]
and this proves the theorem.

**Note 3.** Starting from the function \( f' \) we can similarly show that
\[
\rho_{p,q}(f', g) = \rho_{p,q}(f', g').
\]
Thus we have the following theorem.

**Theorem 6.** Let \( f \) be meromorphic and \( g \) be entire transcendental having the property \((A)\) then
\[
\rho_{p,q}(f', g) = \rho_{p,q}(f', g') = \rho_{p,q}(f', g') = \rho_{p,q}(f, g).
\]

**Note 4.** If one or more of the above expression is infinite, then a slight modification of the proof is needed.

**5. Finiteness of \( \rho_{p,q}(f, g) \)**

In this section we show that the relative \((p, q)\) order of \( f \) with respect to \( g \) is not greater than a certain integral \( \alpha \) that involves the Nevanlinna characteristics of both functions. As a consequence, \( \rho_{p,q}(f, g) \) is finite as soon as \( \alpha \) is.
Definition 5. Let \( f \) be meromorphic and \( g \) be entire. Then we define the number \( \alpha \in [0, +\infty] \) as

\[
\alpha = \inf \{ k \geq 0 : \text{the integral} \int_{r_0}^{\infty} \frac{\log^{[p-2]} T_g^{-1} T_f(r)}{(\log^{[q-1]} r)^{k+1}} dr \text{ converges for some } r_0 = r_0(k) > 0 \}.
\]

Theorem 7. If \( \alpha \) is as in Definition 5, then \( \rho_{p,q}(f, g) \leq \alpha \).

To prove the theorem we require the following lemma.

Lemma 8. If \( \int_{r_0}^{\infty} \frac{\log^{[p-2]} T_g^{-1} T_f(r)}{(\log^{[q-1]} r)^{\lambda+1}} dr \) is convergent for some \( r_0 > 0 \), then

\[
\lim_{r \to \infty} \frac{\log^{[p-2]} T_g^{-1} T_f(r)}{(\log^{[q-1]} r)^{\lambda}} = 0, \text{ where } 0 < \lambda < \infty.
\]

Proof. Given \( \epsilon > 0 \), there is a number \( r'(\epsilon) \geq r_0 \) such that

\[
\int_{r}^{\infty} \frac{\log^{[p-2]} T_g^{-1} T_f(t)}{(\log^{[q-1]} t)^{\lambda+1}} dt < \epsilon \text{ whenever } r > r'(\epsilon)
\]

and so

\[
\int_{r}^{\exp^{[q-1]}(2r)} \frac{\log^{[p-2]} T_g^{-1} T_f(t)}{(\log^{[q-1]} t)^{\lambda+1}} dt < \epsilon \text{ for } r > r'(\epsilon).
\]

So \( \frac{\log^{[p-2]} T_g^{-1} T_f(r)}{(\log^{[q-1]} r)^{\lambda}} < 2^{\lambda+1} \epsilon \) and hence

\[
\lim_{r \to \infty} \frac{\log^{[p-2]} T_g^{-1} T_f(r)}{(\log^{[q-1]} r)^{\lambda}} = 0 \text{ and this proves the lemma.}
\]

Proof of Theorem 7. If \( \alpha = \infty \) then the result is trivial. Assume now that \( \alpha \) is finite. Then for arbitrary \( \epsilon \) \( (0 < \epsilon < 1) \), the integral

\[
\int_{r_0}^{\infty} \frac{\log^{[p-2]} T_g^{-1} T_f(r)}{(\log^{[q-1]} r)^{\alpha+\epsilon+1}} dr \text{ is convergent.}
\]
Therefore by Lemma 8
\[
\lim_{r \to \infty} \frac{\log^{[p-2]} T_g^{-1} T_f(r)}{(\log^{[q-1]} r)^{\alpha+\epsilon}} = 0.
\]
This implies that
\[
\frac{\log^{[p-2]} T_g^{-1} T_f(r)}{(\log^{[q-1]} r)^{\alpha+\epsilon}} < \epsilon
\]
for all sufficiently large \( r \).

So \( \log^{[p-1]} T_g^{-1} T_f(r) < \log \epsilon + (\alpha + \epsilon) \log^{[q]} r \)

i.e.,
\[
\frac{\log^{[p-1]} T_g^{-1} T_f(r)}{\log^{[q]} r} < \alpha + \epsilon
\]
for large \( r \).

Since \( \epsilon > 0 \) is arbitrary, it follows from Theorem 1 that
\[
\rho_{p,q}(f, g) = \limsup_{r \to \infty} \frac{\log^{[p-1]} T_g^{-1} T_f(r)}{\log^{[q]} r} \leq \alpha
\]
as required.

Note 5. Kiselman [5] also introduced a notion of relative order for convex or entire functions, but that does not affect the present paper, because we considered only meromorphic functions and the line of investigations of Kiselman is entirely different from us.

Note 6. We exhibit an example of a meromorphic function whose classical order differs from relative \((p, q)\) order for various values of \( p \) and \( q \). To evaluate the cases, we shall use Theorem 2 several times.

Let \( f(z) = \frac{e^{e^z}}{e^z(z - 1)} \) and \( g(z) = e^z \). Also let \( f_1(z) = \frac{e^{e^z}}{z - 1} \), \( f_2(z) = \frac{1}{e^z} \), \( \phi_1(z) = e^{e^z} \) and \( \phi_2(z) = \frac{1}{z - 1} \). Then we have \( T(r, g) = \frac{r}{\pi} \), \( T(r, \phi_2) = \log r \) and \( T(r, \phi_1) \sim \frac{Ae^r}{r^{p/2}} \), where \( A \) is a finite constant. Now

\[
\rho_{p,q}(\phi_1, g) = \limsup_{r \to \infty} \frac{\log^{[p-1]} \frac{Ae^r}{r^{p/2}}}{\log^{[q]} r} = \begin{cases} 
\infty, & p = q + 1 \\
1, & p = q + 2 \\
0, & p > q + 2.
\end{cases}
\]

For \( p \geq q + 1 \),
\[
\rho_{p,q}(\phi_2, g) = \limsup_{r \to \infty} \frac{\log^{[p-1]} \pi \log r}{\log^{[q]} r} = 0.
\]
Thus

\[ \rho_{p,q}(f_1, g) = \begin{cases} 
\infty, & p = q + 1 \\
1, & p = q + 2 \\
0, & p > q + 2. 
\end{cases} \]

Also

\[ \rho_{p,q}(f_2, g) = \limsup_{r \to \infty} \frac{\log^{[p-1]} \pi r}{\log[g] r} \]
\[ = 1, \quad p = q + 1 \]
\[ = 0, \quad p > q + 1. \]

Hence

\[ \rho_{p,q}(f, g) = \begin{cases} 
\infty, & p = q + 1 \\
1, & p = q + 2 \\
0, & p > q + 2. 
\end{cases} \]

To evaluate the classical order of \( f \), we note that \( \rho(\phi_1) = +\infty \), \( \rho(\phi_2) = 0 \) and so \( \rho(f_1) = +\infty \). Also \( \rho(f_2) = 1 \). Thus using Theorem 5([6]) (taking \( g(z) = e^z \)) we obtain \( \rho(f) = \max\{\rho(f_1), \rho(f_2)\} = +\infty \).

**Note 7.** If \( \phi \) and \( \psi \) are entire functions and \( f = \frac{\phi}{\psi} \) then the relative orders of entire \( \phi \) and \( \psi \) and meromorphic \( f \) are connected by the relation \( \rho_g(f) \leq \max\{\rho_g(\phi), \rho_g(\psi)\} \), where \( \rho_g \) denote the relative order with respect to \( g \) {see [2]}.

**Note 8.** If \( f(z) \) is a meromorphic function and \( \rho_{p,q}(f, g) \) is the relative order with respect to a non exponential entire function, we like to investigate in future how \( \rho_{p,q}(f, g) \) is connected with the distribution of zeros and poles of \( f(z) \) taking into account the Nevanlinna characteristic function.

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References


Department of Mathematics, Visva-Bharati University, Santiniketan, West Bengal-731235, India.
E-mail: dibyendu192@rediffmail.com

Ratneswarbati Netaji High School, Ratneswarbati, Paschim-Medinipur, West Bengal-721212, India.