ON LEFT DERIVATIONS OF $BCI$-ALGEBRAS

BY

HAMZA A. S. ABUJABAL AND NORA O. AL-SHEHRI

Abstract. In the present paper, we introduce the notion of left derivation of a $BCI$-algebra and investigate some related properties. A condition for left derivation to be regular is given. Finally, we give a characterization of a $p$-semisimple $BCI$-algebra which admits left derivation.

1. Introduction

In [3], Y. B. Jun and X. L. Xin applied the notion of derivation in ring and near-ring theory to $BCI$-algebras, and they also introduced a new concept called a regular derivation in $BCI$-algebras. They investigated some of its properties, defined a $d$-invariant ideal and gave conditions for an ideal to be $d$-invariant. In non-commutative rings, the notion of derivations is extended to $\alpha$-derivations, left derivations and central derivations. The properties of $\alpha$-derivations and central derivations were discussed in several papers with respect to the ring structures. For left derivations, M. Brešar and J. Vukman [2] used them to give some results in prime and semi-prime rings. For skew polynomial rings, all left derivations are obtained in a similar way to a polynomial rings (see A. Nakajima and M. Sapanci [8]). In [10], J. Zhan and Y. L. Liu introduced the notion of $f$-derivations of $BCI$-algebras. The objective of this paper is to define left derivation on $BCI$-algebras and then investigate a regular left derivations. Finally, we study left derivations on $p$-semisimple $BCI$-algebras.
2. Preliminaries

Let $X$ be a non-empty set with a binary operation $*$ and a constant 0. The system $(X, *, 0)$ is called a BCI-algebra, if it satisfies the following axioms for all $x, y, z \in X$:

- **BCI-1** $(x * y) * (x * z) * (z * y) = 0$,
- **BCI-2** $(x * (x * y)) * y = 0$,
- **BCI-3** $x * x = 0$,
- **BCI-4** $x * y = 0$ and $y * x = 0$ imply $x = y$.

Define a binary relation $\leq$ on $X$ by putting $x \leq y$ if and only if $x * y = 0$. Then the system $(X, *, 0)$ is a partially ordered set. A BCI-algebra $X$ satisfying $0 \leq x$ for all $x \in X$, is called BCK-algebra. A non-empty subset $I$ of a BCI-algebra $X$ is said to be an ideal of $X$ if it satisfies for all $x, y \in X$:

1. $0 \in I$,
2. $x * y \in I$ and $y \in I$ imply $x \in I$.

Any ideal $I$ has the property $y \in I$ and $x \leq y$ imply $x \in I$.

In any BCI-algebra $X$, the following properties hold for all $x, y, z \in X$:

1. $x * 0 = x$.
2. $(x * y) * z = (x * z) * y$.
3. $0 * (x * y) = (0 * x) * (0 * y)$.
4. $x * (x * y) = x * y$.
5. $((x * z) * (y * z)) * (x * y) = 0$.
6. $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$.
7. $x * 0 = 0$ implies $x = 0$.

For a BCI-algebra $X$, we denote by $X_+ = \{x \in X \mid 0 \leq x\}$, the BCK-part of $X$ and by $G(X) = \{x \in X \mid 0 * x = x\}$, the BCI-G-part of $X$. If $X_+ = \{0\}$, then $X$ is called a p-semisimple BCI-algebra. In a p-semisimple BCI-algebra $X$, the following hold for all $x, y, z \in X$:

8. $(x * z) * (y * z) = x * y$.
9. $0 * (0 * x) = x$.
10. $x * (0 * y) = y * (0 * x)$.
11. $x * y = 0$ implies $x = y$. 
(12) \(x * a = x * b\) implies \(a = b\).

(13) \(a * x = b * x\) implies \(a = b\).

(14) \(a * (a * x) = x\).

Let \(X\) be a \(p\)-semisimple \(BCI\)-algebra. We define addition \(+\) as \(x + y = x * (0 * y)\), for all \(x, y \in X\). Then \((X, +)\) be an abelian group with identity 0 and \(x - y = x * y\). Conversely, let \((X, +)\) be an abelian group with identity 0 and \(x - y = x * y\). Then \(X\) is a \(p\)-semisimple \(BCI\)-algebra and \(x + y = x * (0 * y)\), for all \(x, y \in X\) (see [5]). We denote \(x \land y = y * (y * x)\), \(0 * (0 * x) = a_x\), and

\[
L_P(X) = \{a \in X \mid x * a = 0 \text{ implies } x = a, \forall x \in X\}.
\]

For any \(x \in X\), \(V(a) = \{a \in X \mid a * x = 0\}\) is called the branch of \(X\) with respect to \(a\). We have \(x * y \in V(a * b)\), whenever \(x \in V(a)\) and \(y \in V(b)\), for all \(x, y \in X\) and all \(a, b \in L_P(X)\). Note that \(L_P(X) = \{x \in X \mid a_x = x\}\) which is the \(p\)-semisimple part of \(X\), and \(X\) is a \(p\)-semisimple \(BCI\)-algebra if and only if \(L_P(X) = X\). We note that \(a_x \in L_P(X)\), for \(0 * (0 * a_x) = a_x\), which implies that \(a_x * y \in L_P(X)\), for all \(y \in X\). It is clear that \(G(X) \subseteq L_P(X)\) and \(x * (x * a) = a\) and \(a * x \in L_P(X)\), for all \(a \in L_P(X)\) and all \(x \in X\). For more details, we refer to [1, 4, 7, 9, 11].

**Definition 2.1.** ([6]) A \(BCI\)-algebra \(X\) is said to be commutative if \(x \land y = y \land x\), for all \(x, y \in X\).

**Definition 2.2.** ([3]) Let \(X\) be a \(BCI\)-algebra. By a \((l, r)\)-derivation of \(X\), we mean a self map \(d\) of \(X\) satisfying the identity

\[
d(x * y) = (d(x) * y) \land (x * d(y)), \text{ for all } x, y \in X.
\]

If \(X\) satisfies the identity

\[
d(x * y) = (x * d(y)) \land (d(x) * y), \text{ for all } x, y \in X,
\]

then we say that \(d\) is a \((r, l)\)-derivation of \(X\).

Moreover, if \(d\) is both a \((l, r)\)-derivation and \((r, l)\)-derivation of \(X\), we say that \(d\) is a derivation of \(X\).
Definition 2.3. ([3]) A self-map $d$ of a $BCI$-algebra $X$ is said to be regular if $d(0) = 0$.

Definition 2.4. ([3]) Let $d$ be a self-map of a $BCI$-algebra $X$. An ideal $A$ of $X$ is said to be $d$-invariant, if $d(A) = A$.

3. Left Derivations

In this section, we define the left derivations.

Definition 3.1. Let $X$ be a $BCI$-algebra. By a left derivation of $X$, we mean a self-map $D$ of $X$ satisfying

$$D(x * y) = (x * D(y)) \land (y * D(x)),\text{ for all } x, y \in X.$$ 

Example 3.2. Let $X = \{0, 1, 2\}$ be a $BCI$-algebra with Cayley table defined by

<table>
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<tr>
<th>*</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

Define a map $D : X \rightarrow X$ by

$$D(X) = \begin{cases} 
2, & \text{if } x = 0, 1 \\
0, & \text{if } x = 2.
\end{cases}$$

Then it is easily checked that $D$ is a left derivation of $X$.

Proposition 3.3. Let $D$ be a left derivation of a $BCI$-algebra $X$. Then for all $x, y \in X$, we have

1. $x * D(x) = y * D(y)$.
2. $D(x) = a_{D(x) \land x}$.
3. $D(x) = D(x) \land x$.
4. $D(x) \in L_P(X)$. 

Proof. (1) Let $x, y \in X$. Then
\[
D(0) = D(x \cdot x) = (x \cdot D(x)) \land (x \cdot D(x)) = x \cdot D(x).
\]
Similarly, $D(0) = y \cdot D(y)$. So, $x \cdot D(x) = y \cdot D(y)$.

(2) Let $x \in X$. Then
\[
D(x) = D(x \cdot 0)
= (x \cdot D(0)) \land (0 \cdot D(x))
= (0 \cdot D(x)) \cdot ((0 \cdot D(x)) \cdot (x \cdot D(0)))
\leq 0 \cdot (0 \cdot (x \cdot D(0)))
= 0 \cdot (0 \cdot (x \cdot (x \cdot D(x))))
= 0 \cdot (0 \cdot (D(x) \land x))
= a_{D(x) \land x}.
\]
Thus $D(x) \leq a_{D(x) \land x}$. But
\[
a_{D(x) \land x} = 0 \cdot (0 \cdot (D(x) \land x)) \leq D(x) \land x \leq D(x).
\]
Therefore, $D(x) = a_{D(x) \land x}$.

(3) Let $x \in X$. Then using (2), we have
\[
D(x) = a_{D(x) \land x} \leq D(x) \land x,
\]
but we know that $D(x) \land x \leq D(x)$, and hence (3) holds.

(4) Since $a_x \in L_P(X)$, for all $x \in X$, we get $D(x) \in L_P(X)$ by (2).

Remark 3.4. Proposition 3.3(4) implies that $D(X)$ is a subset of $L_P(X)$.

Proposition 3.5. Let $D$ be a left derivation of a $BCI$-algebra $X$. Then for all $x, y \in X$, we have
(1) $y \cdot (y \cdot D(x)) = D(x)$.
(2) $D(x) \cdot y \in L_P(X)$.

Proposition 3.6. Let $D$ be a left derivation of a $BCI$-algebra $X$. Then
(1) $D(0) \in L_P(X)$.
(2) $D(x) = 0 + D(x)$, for all $x \in X$. 
(3) \( D(x + y) = x + D(y) \), for all \( x, y \in L_P(X) \).

(4) \( D(x) = x \), for all \( x \in X \) if and only if \( D(0) = 0 \).

(5) \( D(x) \in G(X) \), for all \( x \in G(X) \).

**Proof.**

(1) Follows by Proposition 3.3(4).

(2) Let \( x \in X \). From Proposition 3.3(4), we get

\[
D(x) = a_D(x) = 0 \ast (0 \ast D(x)) = 0 + D(x).
\]

(3) Let \( x, y \in L_P(X) \). Then

\[
D(x + y) = D(x \ast (0 \ast y)) \\
= (x \ast D(0 \ast y)) \land ((0 \ast y) \ast D(x)) \\
= ((0 \ast y) \ast D(x)) \ast ((0 \ast y) \ast D(x)) \ast (x \ast D(0 \ast y)) \\
= x \ast D(0 \ast y) \\
= x \ast ((0 \ast D(y)) \land (y \ast D(0))) \\
= x \ast D(0 \ast y) \\
= x \ast (0 \ast D(y)) \\
= x + D(y).
\]

(4) Let \( D(0) = 0 \) and \( x \in X \). Then

\[
D(x) = D(x) \land x = x \ast (x \ast D(x)) = x \ast D(0) = x \ast 0 = x.
\]

Conversely, let \( D(x) = x \), for all \( x \in X \). So it is clear that \( D(0) = 0 \).

(5) Let \( x \in G(X) \). Then \( 0 \ast x = x \) and so

\[
D(x) = D(0 \ast x) \\
= (0 \ast D(x)) \land (x \ast D(0)) \\
= (x \ast D(0)) \ast ((x \ast D(0)) \ast (0 \ast D(x))) \\
= 0 \ast D(x).
\]

This gives \( D(x) \in G(X) \).

**Remark 3.7.** Proposition 3.6(4) shows that a regular left derivation of a BCI-algebra is the identity map. So we have the following:
**Proposition 3.8.** A regular left derivation of a $BCI$-algebra is trivial.

**Remark 3.9.** Proposition 3.6(5) gives that $D(x) \in G(X) \subseteq L_P(X)$.

**Definition 3.10.** An ideal $A$ of a $BCI$-algebra $X$ is said to be $D$-invariant, if $D(A) \subseteq A$.

Now, Proposition 3.8 helps to prove the following theorem.

**Theorem 3.11.** Let $D$ be a left derivation of a $BCI$-algebra $X$. Then $D$ is regular if and only if every ideal of $X$ is $D$-invariant.

**Proof.** Let $D$ be a regular left derivation of a $BCI$-algebra $X$. Then Proposition 3.8 gives that $D(x) = x$, for all $x \in X$. Let $y \in D(A)$, where $A$ is an ideal of $X$. Then $y = D(x)$, for some $x \in A$. Thus

$$y \ast x = D(x) \ast x = x \ast x = 0 \in A.$$  

Then $y \in A$ and $D(A) \subseteq A$. Therefore, $A$ is $D$-invariant.

Conversely, let every ideal of $X$ be $D$-invariant. Then $D(\{0\}) \subseteq \{0\}$, and hence $D(0) = 0$ and $D$ is regular.

Finally, we give a characterization of a left derivation of a $p$-semisimple $BCI$-algebra.

**Proposition 3.12.** Let $D$ be a left derivation of a $p$-semisimple $BCI$-algebra. Then the following hold for all $x, y \in X$:

1. $D(x \ast y) = x \ast D(y)$.
2. $D(x) \ast x = D(y) \ast y$.
3. $D(x) \ast x = y \ast D(y)$.

**Proof.** (1) Let $x, y \in X$. Then

$$D(x \ast y) = (x \ast D(y)) \wedge (y \ast D(x)) = x \ast D(y).$$

(2) We know that

$$(x \ast y) \ast (x \ast D(y)) \leq D(y) \ast y$$

and

$$(y \ast x) \ast (y \ast D(x)) \leq D(x) \ast x.$$
This means that
\[(x \ast y) \ast (x \ast D(y)) \ast (D(y) \ast y) = 0,
\]
and
\[(y \ast x) \ast (y \ast D(x)) \ast (D(x) \ast x) = 0.
\]
So
\[(x \ast y) \ast (x \ast D(y)) \ast (D(y) \ast y) = ((y \ast x) \ast (y \ast D(x))) \ast (D(x) \ast x). \tag{I}
\]
Using Proposition 3.3(1), we get
\[(x \ast y) \ast D(x \ast y) = (y \ast x) \ast D(y \ast x). \tag{II}
\]
By (I), (II) yields
\[(x \ast y) \ast (x \ast D(y)) = (y \ast x) \ast (y \ast D(x)).
\]
Since \(X\) is a \(p\)-semisimple \(BCI\)-algebra. (I) implies that
\[D(x) \ast x = D(y) \ast y.
\]
(3) We have, \(D(0) = x \ast D(x)\). From (2), we get \(D(0) \ast 0 = D(y) \ast y\) or \(D(0) = D(y) \ast y\). So \(D(x) \ast x = y \ast D(y)\).

**Theorem 3.13.** In a \(p\)-semisimple \(BCI\)-algebra \(X\), a self-map \(D\) of \(X\) is left derivation if and only if it is derivation.

**Proof.** Assume that \(D\) is a left derivation of a \(BCI\)-algebra \(X\). First, we show that \(D\) is a \((r, l)\)-derivation of \(X\). Then
\[
D(x \ast y) = x \ast D(y)
= (D(x) \ast y) \ast ((D(x) \ast y) \ast (x \ast D(y)))
= (x \ast D(y)) \land (D(x) \ast y).
\]
Now, we show that $D$ is a $(r, l)$-derivation of $X$. Then

$$D(x * y) = x * D(y)$$

$$= (x * 0) * D(y)$$

$$= (x * (D(0) * D(0))) * D(y)$$

$$= (x * ((x * D(x)) * (D(y) * y))) * D(y)$$

$$= (x * ((x * D(y)) * (D(x) * y))) * D(y)$$

$$= (x * D(y)) * ((x * D(y)) * (D(x) * y))$$

$$= (D(x) * y) \land (x * D(y)).$$

Therefore, $D$ is a derivation of $X$.

Conversely, let $D$ be a derivation of $X$. So it is a $(r, l)$-derivation of $X$. Then

$$D(x * y) = (x * D(y)) \land (D(x) * y)$$

$$= (D(x) * y) * ((D(x) * y) * (x * D(y)))$$

$$= x * D(y) = (y * D(x)) * ((y * D(x)) * (x * D(y)))$$

$$= (x * D(y)) \land (y * D(x)).$$

Hence, $D$ is a left derivation of $X$.

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**References**


Department of Mathematics, Faculty of Science, King Abdulaziz University, P. O. Box 80003, Jeddah, 21589, Saudi Arabia.
E-mail: prof_h_abujabal@yahoo.co

Department of Mathematics, Faculty of Education, Science Sections, P. O. Box 33910, Jeddah, 21458, Saudi Arabia.
E-mail: noooora55@hotmail.com